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A Comparison of the Mean–Variance–Leverage Optimization Model and the Markowitz General Mean–Variance Portfolio Selection Model

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In response to the mean–variance-leverage optimization model (MVL model) proposed in Jacobs and Levy [2013], Markowitz [2013] suggested a broader application of the general mean–variance portfolio selection model (GPSM) (Markowitz [1959], Markowitz and Todd [2000]) that differs from conventional usage of the GPSM.

We developed the MVL model because the mean–variance (MV) optimization inherent in the conventional GPSM does not consider components of risk that are unique to using leverage. These include the risks and costs of margin calls—which can force borrowers to liquidate securities at adverse prices due to illiquidity—losses exceeding the capital invested, and the possibility of bankruptcy.

The MVL model provides practical insights on investors' aversion to leverage, as well as a methodology that is straightforward and ready to implement. In contrast, practical implementation of the broader GPSM, and economic insights it may provide, are dependent on the successful future development of a stochastic margin call model (SMCM).

Markowitz [2013] noted that our MVL model uses part of the apparatus of the conventional GPSM. Like the GPSM, the MVL model allows for shorts, leveraged longs, and constraints on both individual securities and the portfolio. Our joint previous work with Markowitz addressed leverage-constrained GPSM optimization (Jacobs, Levy, and Markowitz [2005, 2006]). There are, however, several aspects of the MVL model that differentiate it from both the conventional and broader GPSM.

LEVERAGE RISK—A THIRD DIMENSION

In Jacobs and Levy [2013], we specified an MVL utility function that is broader than the MV utility function used in the conventional GPSM. We augmented the MV utility function with a leverage-aversion term, which is the third term in the following equation:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L}\sigma_T^2\Lambda^2 \tag{1}$$

where α_P is the expected active return (relative to the benchmark) of the leveraged portfolio; σ_P^2 is the variance of the leveraged portfolio's active return; τ_V is the investor's risk tolerance with respect to the variance of the portfolio's active return, which we will refer to as volatility tolerance; σ_T^2 is the variance of the leveraged portfolio's total return; τ_L is the investor's leverage tolerance; and Λ is the sum of the absolute values of the portfolio holding weights minus 1:[1]

$$\Lambda = \sum_{i=1}^{N}\left|h_i\right| - 1 \tag{2}$$

where h_i is the portfolio holding weight of security i, and N is the number of securities in the selection universe. The active weight, x_i, of security i is equal to h_i minus its benchmark weight, b_i:

$$x_i = h_i - b_i \tag{3}$$

The MVL model extends the MV model to incorporate leverage risk as a third dimension. Similarly, Markowitz [2013] introduced a third dimension to the GPSM in the form of short-run portfolio variance, V^S. It is important to note that without a third dimension, the unique risks of leverage would not be represented.

The MVL leverage-aversion term in Equation (1) is a proxy for leverage risk.[2] This assumes that the risks of leverage rise with the product of the variance of the leveraged portfolio's total return and the square of the portfolio's leverage. By multiplying the square of the portfolio leverage by the variance of total return, we obtain a measure of the leverage risk's severity. For instance, leveraging highly volatile stocks entails more leverage risk (or margin-call risk) than leveraging stable stocks.

Also, by using the variance of total return (the square of the standard deviation of total return) to represent portfolio volatility, and by squaring leverage, the model recognizes that margin-call risk increases more rapidly as volatility and/or leverage increases. An advantage of the MVL model proxying leverage risk, rather than modeling margin calls explicitly, is that the MVL's leverage-aversion term is easily calculated. It requires only expected total portfolio volatility, the level of portfolio leverage, and the investor's leverage tolerance.

In contrast, Markowitz's expanded model requires the development of an SMCM, to be used along with measures of short-run portfolio variance, V^S, and long-run portfolio variance, V^L. These portfolio-variance measures require defining the periods referred to as short run and long run. The SMCM would be used to calculate 1) the probability that there will be one or more margin calls on a portfolio; 2) the portfolio return variance conditional on the occurrence of one or more margin calls; and 3) the reduction in the portfolio's expected return conditional on the occurrence of one or more margin calls.

To accurately incorporate all these risks is a formidable problem that has yet to be solved. As Markowitz [2013] noted, only when one has developed a good SMCM can the broader GPSM find realistic efficient portfolios.

QUARTIC VERSUS QUADRATIC OPTIMIZATION

The MVL approach differs from both GPSM approaches in the computation required. The GPSM is a quadratic mathematical problem (Markowitz [1959]). All terms in the utility function are linear or quadratic. A linear term is directly proportional to the active weight variable x_i. A quadratic term is proportional to the square of the active-weight variable x_i, including second-order cross-product terms. The application of GPSM as proposed in Markowitz [2013] would require using a quadratic solver repeatedly to create many mean–variance efficient portfolios, then computing the adjusted mean and variance of these portfolios by running each portfolio through the SMCM to make adjustments for margin-call risk.

The MVL model, in contrast, is a quartic mathematical problem. The leverage-aversion term in the MVL utility function (representing the product of the square of the standard deviation of the leveraged portfolio's total return and the square of the portfolio's leverage) is quartic in the active weight variable x_i, including fourth-order cross-product terms. Hence, the MVL cannot be solved directly with quadratic optimization. A solution method is fixed-point iteration that applies a quadratic solver iteratively in order to provide the optimal portfolio (Jacobs and Levy [2012, 2013]).

PRACTICAL INSIGHTS FROM THE MVL OPTIMIZATION MODEL

The broader GPSM and the MVL model differ in the underlying economic intuition and insights that they provide. The GPSM approach proposed by Markowitz [2013] is unable to yield practical economic insights until an accurate SMCM has been developed. In contrast, the MVL approach yields practical insights that can help investors understand the importance of leverage in selecting optimal portfolios.

A comparison of the MVL model and the conventional GPSM shows that the MV model is a special case of the MVL model. As the investor's tolerance for the unique risks of leverage approaches zero, the investor has an infinite aversion to leverage and the optimizer forces the portfolio's leverage level to zero. Because there is no leverage present, the MVL model reduces to the traditional long-only MV model. At the other extreme, as the investor's tolerance for the unique risks of leverage approaches infinity, the investor has no aversion to leverage, the leverage term in the MVL model is multiplied by zero leverage aversion, and that term drops out of the MVL utility function. Again, the MVL model reduces to the MV model.

The MV model, used with a constraint enforcing zero leverage, therefore implies that the investor has an infinite aversion to the unique risks of leverage, or zero leverage tolerance. Used without a leverage constraint, the MV model implies that the investor has zero aversion to the unique risks of leverage, or infinite leverage tolerance. Note that, although we observe zero leverage tolerance in practice—some investors are averse to any borrowing—infinite leverage tolerance seems contrary to investor behavior, because it can give rise to extreme levels of leverage, in the absence of upper bounds on individual security holdings (Jacobs and Levy [2013, 2014]).

To avoid excessive leverage, the common practice today is to constrain it at some level.[3] For an investor who is averse to leverage, using the conventional mean–variance utility function and optimizing with a leverage constraint is unlikely to lead to the portfolio offering the highest utility. This is because a leverage constraint denies the investor the ability to balance the economic tradeoffs between expected portfolio return, portfolio volatility risk, and portfolio leverage risk (Jacobs and Levy [2014]).

Using the MVL model, there is a different MV efficient frontier for any given level of investor leverage tolerance. Rather than one conventional MV efficient frontier, there are numerous frontiers, which fan out into an efficient region (Jacobs and Levy [2013]). For a particular level of leverage tolerance, there exists a unique MV efficient frontier within the efficient region. Investors select optimal portfolios from the efficient frontier that correspond to their preferred leverage tolerances. Investors may prefer a lower frontier to a higher frontier because they may have a more moderate level of leverage tolerance, even though a lower frontier offers a lower level of expected return at each expected volatility level than does a higher frontier.

It may appear that this choice of a lower frontier contradicts the basic tenets of mean–variance portfolio theory. However, the investor's preference for a lower frontier, despite its lower expected returns, reflects the investor's aversion to the unique risks associated with the higher frontier's higher leverage. The volatility-averse MV investor accepts a lower expected return in exchange for less volatility risk, and the leverage-averse MVL investor accepts a lower expected return in exchange for less volatility risk and less leverage risk. Put simply, leverage aversion affects portfolio choice.

CONCLUSION

The MVL model provides many practical insights, and implementation for portfolio selection is straightforward. In contrast, practical use of the broader GPSM, as suggested by Markowitz, is dependent on the successful future development of an SMCM.

The MVL model allows investors to consider both volatility tolerance and leverage tolerance in selecting optimal portfolios from either a three-dimensional MVL efficient surface (Jacobs and Levy [2012, 2014]), or a two-dimensional mean–variance-efficient region (Jacobs and Levy [2013]). The MVL formulation provides a formal model that has intuitive appeal.

Relying on leverage constraints with a conventional GPSM, as is commonly done today, is unlikely to lead to the portfolio offering a leverage-averse investor the highest utility. However, investors can use the MVL model to find optimal portfolios that balance expected return, volatility risk, and leverage risk.

ENDNOTES

[1]Leverage is measured in excess of 1, that is, in excess of 100% of net capital.

[2]We use the terms "tolerance" and "aversion" with the understanding that they are the inverse of each other.

[3]There may be some absolute leverage level that an MV investor is required not to exceed. A certain level of leverage can be specified as an equality constraint or, alternatively, leverage can be constrained not to exceed a certain level, as an inequality constraint. A leverage constraint can also be imposed in MVL optimization.

REFERENCES

Jacobs, B., and K. Levy. "Leverage Aversion and Portfolio Optimality." *Financial Analysts Journal*, Vol. 68, No. 5 (2012), pp. 89-94.

——. "Leverage Aversion, Efficient Frontiers, and the Efficient Region." *The Journal of Portfolio Management*, Vol. 39, No. 3 (2013), pp. 54-64.

——. "Traditional Optimization Is Not Optimal for Leverage-Averse Investors." *The Journal of Portfolio Management*, Vol. 40, No. 2, forthcoming 2014.

Jacobs, B., K. Levy, and H. Markowitz. "Portfolio Optimization with Factors, Scenarios, and Realistic Short Positions." *Operations Research*, Vol. 53, No. 4 (2005), pp. 586-599.

——. "Trimability and Fast Optimization of Long-Short Portfolios." *Financial Analysts Journal*, Vol. 62, No. 2 (2006), pp. 36-46.

Markowitz, H. *Portfolio Selection: Efficient Diversification of Investments*. New Haven, CT: Yale University Press, 1959; 2nd ed. Cambridge, MA: Blackwell Publishing, 1991.

——. "How to Represent Mark-to-Market Possibilities with the General Portfolio Selection Model." *The Journal of Portfolio Management*, Vol. 39, No. 4 (2013), pp. 1-3.

Markowitz, H., and G. Peter Todd. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. New Hope, PA: Frank J. Fabozzi Associates, 2000.

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