

# Leverage Aversion and Portfolio Optimality

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*Portfolio volatility is the only source of risk in mean–variance optimality, but it fails to capture all the risks faced by leveraged portfolios. These risks include the possibility of margin calls and forced liquidations at adverse prices and losses beyond the capital invested. To recognize these risks, the authors incorporated leverage aversion into the optimization process and examined the effects of volatility and leverage aversion on optimal long–short portfolios.*

**A** portfolio with borrowing (for short or long positions) differs in a fundamental way from a portfolio without borrowing. For the latter, losses cannot exceed the invested capital. A portfolio with leverage can sustain losses beyond the capital invested. For a portfolio leveraged via short sales, losses can theoretically be unlimited (because asset prices can rise without limit).

Furthermore, losses on an unleveraged portfolio may lead to the sale of portfolio assets, but such sales are not forced. A portfolio with leverage can be subject to forced sales (deleveraging)—even absent explicit constraints on leverage or losses that threaten to eradicate portfolio capital—because the lender facilitating the leverage may increase collateral demands on the borrower as losses mount in order to ensure that its own risk limits and capital requirements are not endangered. Meeting such margin calls may require (in the absence of an infusion of additional capital) a forced liquidation of assets. All too frequently, such sales, or short covers, take place when market prices are already adverse, further exacerbating price movements, widening losses on the portfolio, and straining liquidity in the market. This was the case with Long-Term Capital Management in 1998 (see Jacobs and Levy 2005), with the Goldman Sachs Global Equity Opportunities Fund and a number of other “quant” funds in the summer of 2007, and with the collapses of Bear Stearns and Lehman Brothers in 2008 (Jacobs 2009).

Leverage thus introduces its own set of risks, which are not incurred by an unleveraged portfolio. Consideration of such risks is likely to limit the amount of leverage an investor is willing to bear,

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even when there are no formal constraints on leverage (such as Federal Reserve Board Regulation T). For instance, in an unconstrained mean–variance optimization framework, leverage increases monotonically with risk tolerance. Subject to the investor’s risk tolerance, such optimization could lead to a portfolio with extremely high leverage. An investor may very well balk at such high leverage, either because of explicit consideration of the probability of margin calls, forced liquidation, and the cost of bankruptcy or simply because of an instinctive sense that too much leverage is bad.

Portfolio volatility, which is the only source of risk in mean–variance optimality, fails to capture all the risks faced by leveraged portfolios. We believe that risk tolerance (or its inverse, risk aversion) is multidimensional and that both volatility aversion and leverage aversion should play a role in selecting an optimal portfolio. Leverage aversion places at least implicit restraints on the investor’s appetite for leverage (and on potential lenders’ willingness to underwrite it) and can explain the difference between the apparent mean–variance optimality of leveraged portfolios and their suboptimal nature from the investor’s perspective.

Therefore, we propose that mean–variance utility (see Markowitz 1952) be augmented to incorporate leverage aversion. When leverage aversion is included in the investor’s utility function, optimization yields portfolios that are much more consistent with the level of leverage that most investors would find acceptable.

In this study, we examined the role of leverage in the context of long–short portfolios—specifically, enhanced active equity (EAE) portfolios. An EAE portfolio maintains a 100% exposure to an underlying stock market benchmark while relaxing the long-only constraint to allow for short sales equal to some percentage of capital and for use of the short-sale proceeds to buy additional securities long (see

Jacobs and Levy 2007a). For example, short sales equal to 30% of capital provide for a 30% expansion of long positions, giving rise to an enhanced active 130–30 portfolio. In such a portfolio, leverage is attained via short selling. We looked at what amount of short selling as a percentage of capital (the enhancement) is optimal given various levels of investor leverage aversion.<sup>1</sup>

A number of researchers have analyzed the optimal level of enhancement for long–short portfolios, including Johnson, Kahn, and Petrich (2007); Sorensen, Hua, and Qian (2007); and Clarke, de Silva, Sapra, and Thorley (2008). Johnson, Kahn, and Petrich showed that the optimal value of “gearing,” a concept akin to enhancement, is a linearly increasing function of risk tolerance, forecast accuracy, and the number of securities in the portfolio; it is inversely proportional to security volatility. The authors also demonstrated that expected alpha, risk, and gearing must be considered simultaneously; attempting to specify the optimal level of any two of these quantities independently runs the risk of pushing the portfolio away from an optimal relationship among the three. They deepened their analysis by considering the effect of suboptimal gearing on the transfer coefficient. (This metric relates the realized information ratio to the unconstrained information ratio and proxies for the efficiency with which information from the analytical process is transferred to the actual portfolio.<sup>2</sup>) When such costs are considered, the transfer coefficient shows a well-defined peak as a function of risk with a given level of gearing or gearing with a given level of risk.

In examining optimization that explicitly considers the benchmark underlying an EAE portfolio, Clarke, de Silva, Sapra, and Thorley (2008) developed a number of propositions. In particular, the optimal amount of short selling (enhancement) increases with the targeted level of active risk, with higher correlations between security returns, with the number of securities in the benchmark, with forecast accuracy, and with the degree to which the benchmark is concentrated in large-cap names. It decreases with individual security risk. Borrowing costs and general portfolio operating costs, such as transaction costs, reduce the optimal level of leverage.

We incorporate a leverage tolerance term (capturing investor leverage aversion) into the investor’s utility function, in addition to the standard volatility tolerance term. We then examine the effects of this augmented utility function on the amount of leverage in the optimal portfolio. We find that leverage tolerance has a significant effect on the optimal level of enhancement, even

in the absence of cost considerations, and suggest the inclusion of leverage tolerance in investor utility functions.<sup>3</sup>

## Optimal Enhancement with Leverage Aversion

The conventional mean–variance portfolio optimization problem is concerned with choosing a portfolio that maximizes the investor’s utility, expressed as

$$U = \alpha_P - \frac{1}{2\tau_V} \sigma_P^2, \quad (1)$$

where  $\alpha_P$  is the portfolio’s expected active return,  $\sigma_P^2$  is the variance of the portfolio’s active return, and  $\tau_V$  is the investor’s risk tolerance, where risk tolerance is with respect to the portfolio’s active return volatility. We use the terms *tolerance* and *aversion* with the understanding that these two quantities are the inverse of each other.

We define portfolio leverage as

$$\Lambda = \sum_{i=1}^N |h_i| - 1, \quad (2)$$

where  $h_i$  is the weight of the portfolio’s holding of security  $i$  and  $N$  is the number of securities in the selection universe. The portfolio’s enhancement is  $E = \Lambda/2$ . For example, consider an EAE portfolio with 130% of capital long and 30% short. The leverage (the sum of the absolute values of the weights minus 1) is 0.6, and the enhancement is 0.3, or 30%.

Leverage could be included in the utility function in several ways. For example, it could be included as a linear term or a squared term. We prefer to use a squared term because we regard leverage as a risk component, which should therefore have a form similar to that of the other risk component in the utility function, volatility. We propose extending the utility in Equation 1 to the following augmented utility function, which includes leverage tolerance  $\tau_L$  as well as volatility tolerance  $\tau_V$ :

$$U = \alpha_P - \frac{1}{2\tau_V} \sigma_P^2 - \frac{c}{2\tau_L} \Lambda^2. \quad (3)$$

The coefficient of  $\Lambda^2$  includes a scaling constant  $c$  to give a similar order of magnitude to the two separate risk terms. Equation 3 can be maximized for any nonzero pair  $(\tau_V, \tau_L)$ .<sup>4</sup> The augmented utility function reduces to the mean–variance utility function as the investor’s leverage tolerance,  $\tau_L$ , increases without limit. The augmented utility function also reduces to the mean–variance utility function as the investor’s leverage tolerance

approaches zero, resulting in a “long-only” portfolio, because the investor avoids any leverage,  $\Lambda$ .

Although we assume that investors have the same aversion to leveraged long positions as they do to short positions, this assumption may not be the case in practice because short positions have potentially unlimited liability and are susceptible to short squeezes. One could model the aversion to long and short positions asymmetrically. Because doing so would complicate the algebra, for simplicity we will continue to use a common leverage tolerance.

In optimizing the utility in Equation 3, we use active security weights and active security returns. The active weight,  $x_i$ , of security  $i$  is equal to its holding weight,  $h_i$ , minus its benchmark weight,  $b_i$ :

$$x_i = h_i - b_i \quad (4)$$

The active return,  $r_i$ , of security  $i$  is the difference between the security's return and the benchmark's return. The expected active return for security  $i$  is  $\alpha_i$ . The portfolio's expected active return is then

$$\alpha_P = \sum_{i=1}^N \alpha_i x_i \quad (5)$$

If  $\sigma_{ij}$  is the covariance between the active returns of securities  $i$  and  $j$ , then the variance of the portfolio's active return is

$$\sigma_P^2 = \sum_{i=1}^N \sum_{j=1}^N x_i \sigma_{ij} x_j \quad (6)$$

Using Equations 5 and 6, the utility function in Equation 1 is equivalent to the following:

$$U = \sum_{i=1}^N \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^N \sum_{j=1}^N x_i \sigma_{ij} x_j \quad (7)$$

Equation 3 provides a utility function that explicitly considers leverage aversion. Using Equations 2, 4, 5, and 6, Equation 3 can be expressed as

$$U = \sum_{i=1}^N \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^N \sum_{j=1}^N x_i \sigma_{ij} x_j - \frac{c}{2\tau_L} \left( \sum_{i=1}^N |b_i + x_i| - 1 \right)^2 \quad (8)$$

The standard constraint set for an EAE portfolio is

$$\sum_{i=1}^N h_i = 1 \quad (9)$$

and

$$\sum_{i=1}^N h_i \beta_i = 1 \quad (10)$$

Equation 9 is the full-investment (net longs minus shorts) constraint, which requires that the sum of the holding weights equal 1. Equation 10 is the beta constraint, which requires that the portfolio's beta equal 1. Using active weights, these constraints are expressed as

$$\sum_{i=1}^N x_i = 0 \quad (11)$$

and

$$\sum_{i=1}^N x_i \beta_i = 0 \quad (12)$$

## An Example with Leverage Aversion

We utilize Equation 8, the utility function to be maximized subject to the constraint Equations 11 and 12, to develop an empirical example of how various combinations of  $\tau_V$  and  $\tau_L$  affect optimal levels of enhancement. The required inputs for Equation 8 include expected active return,  $\alpha_i$ , covariance,  $\sigma_{ij}$ , and the scaling constant  $c$ . To estimate these, we used daily return data for the constituent stocks in the S&P 100 Index over the two years (505 trading days) ending on 30 September 2011.<sup>5</sup>

To produce estimates for securities' expected active returns, given that investors have imperfect foresight of future returns, we used the following transformation:

$$\alpha_i = s \left( s \bar{r}_i + \sqrt{1 - s^2} \sigma_r \varepsilon_i \right), \quad (13)$$

where  $s$  is the skill, or information coefficient (that is, the correlation between predicted and actual active returns),  $\bar{r}_i$  is the average daily active return for each security  $i$ ,  $\sigma_r$  is the standard deviation of the average daily active returns across the securities, and  $\varepsilon_i$  is a standard normal random variable that is independent of  $\bar{r}_i$ .<sup>6</sup> We set  $s$  equal to 0.1, a value representing a manager with strong insight.

We used the actual daily active returns to compute the covariances used in Equation 8. Predicting covariances is relatively easier than predicting returns because covariances tend to be more stable. Hence, for this example, we assumed that the future covariances were known.

For the scaling constant  $c$ , we used the average, across all the securities, of the variance of each stock's daily active returns.<sup>7</sup>

We found EAE portfolios that maximize the utility function represented by Equation 8 for a range of volatility and leverage tolerance pairs ( $\tau_V$ ,  $\tau_L$ ), subject to the constraints that set the portfolio's

market exposure and beta equal to 1. Securities' betas were estimated from the daily return data using the single index market model with the S&P 100 Index.<sup>8</sup> In addition, we constrained each security's active weight to be between -10% and +10%.

The constraint set does not impose any given level of enhancement. The level of enhancement (or leverage) that emerges is therefore the optimal one for an investor's particular choice of  $(\tau_V, \tau_L)$ . We chose  $100 \times 100$  pairs of values of  $(\tau_V, \tau_L)$  to cover the range  $[0.001, 2]$  for a total of 10,000 optimizations.<sup>9</sup>

The optimal enhancement obtained as a function of  $\tau_V$  and  $\tau_L$  for this example is shown in the surface plot of **Figure 1**.<sup>10</sup> The z-axis represents the sum of the short weights (i.e., the optimal enhancement). Thus, for example, an indicated enhancement of 30% corresponds to a 130-30 portfolio. Figure 1 shows that as volatility tolerance increases, the optimal enhancement increases, rapidly at first. As leverage tolerance increases, the optimal enhancement increases at a slower rate. The optimal enhancement levels off more slowly in the case of leverage tolerance than in the case of volatility tolerance.

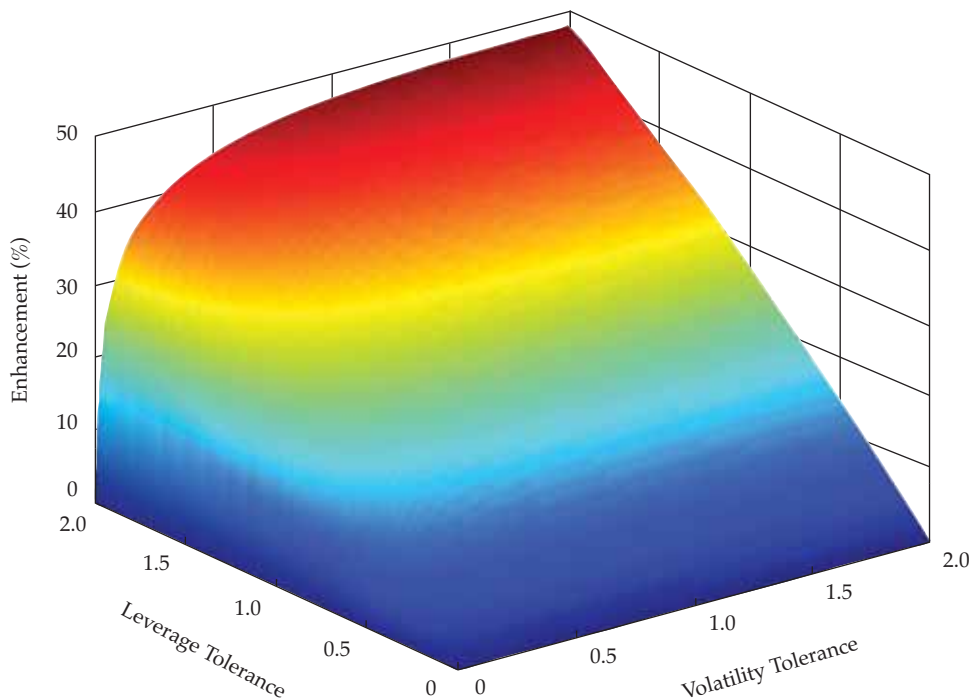
Some investors may have zero leverage tolerance, in which case their portfolios lie along the

volatility tolerance axis. These portfolios necessarily have unlevered long positions and no short positions ("long-only") and hence 0% enhancement. Some investors may have zero volatility tolerance, in which case their portfolios lie along the leverage tolerance axis. These portfolios necessarily have no active return volatility and hence hold benchmark weights in each security ("index fund").

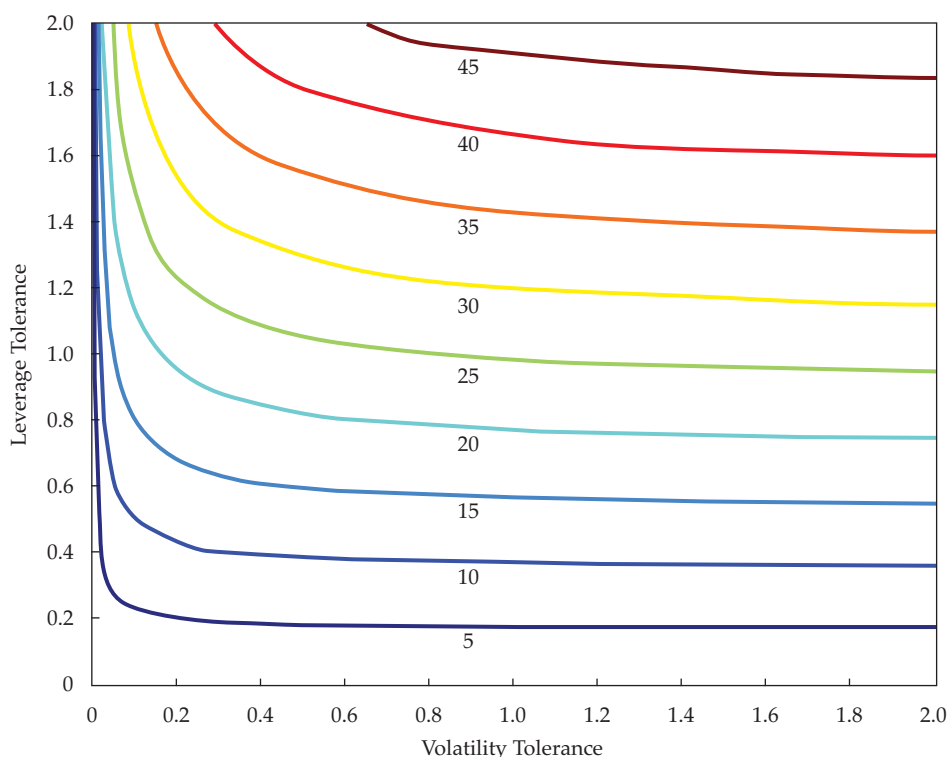
Giving an alternative view of the optimal enhancement, **Figure 2** shows a contour plot of the surface from Figure 1. Each contour line is labeled with its enhancement level. For instance, the yellow contour line in Figure 2 shows all the portfolios with a 30% enhancement corresponding to the portfolios in the yellow region in Figure 1. For an investor with a volatility tolerance of 1 and a leverage tolerance of 1, this example shows the optimal enhancement to be about 25%.

The contour lines show that the optimal enhancement increases with leverage tolerance, but it is approximately independent of volatility tolerance if the latter is large enough. For example, for a leverage tolerance of 0.4 and for all values of volatility tolerance above 0.4, the optimal enhancement is about 10%.

**Figure 1. Optimal Enhancement Surface for Various Combinations of Volatility and Leverage Tolerance**



**Figure 2. Contour Map of Optimal Enhancement for Various Combinations of Volatility and Leverage Tolerance**



The optimal level of enhancement is highly dependent upon the investor's particular level of leverage tolerance. For a volatility tolerance of 1 and leverage tolerances between 0 and 2, the optimal level of enhancement ranges from less than 5% to more than 45%.

## Conclusion

A leveraged portfolio may be subject to margin calls and forced liquidations at adverse prices; furthermore, it can sustain losses beyond the capital invested. Leverage thus introduces sources of risk that are not captured by conventional mean-variance optimization. In order to recognize these risks, we proposed that portfolio optimization include a measure of leverage aversion as well as

volatility aversion. We optimized long-short portfolios and showed their optimal levels of enhancement given various levels of leverage aversion. When leverage aversion is added to the investor's utility function, optimization yields portfolios that are more consistent with the levels of leverage generally seen in practice. Further, the explicit recognition of leverage aversion by investors might curtail some of the outsized levels of leverage and consequent market disruptions that have been experienced in recent years.

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*This article qualifies for 1 CE credit.*

## Notes

1. Many hedge funds use much higher leverage than 130–30 portfolios. But investors do not typically put their wealth into one strategy, investing instead in several strategies that may have low correlations with one another. So, even if investors put some of their wealth into a high-leverage strategy, their overall leverage may be lower. Still, investors should consider their overall exposure to leverage in their asset allocation decision.
2. See Clarke, de Silva, and Thorley (2002) for a definition and discussion of the transfer coefficient.
3. For a discussion of how cost considerations lead to EAE portfolios that are compact and less leveraged, see Jacobs and Levy (2007b).
4. Specifying portfolio volatility and leverage as two independent, additive terms, as in Equation 3, may be an oversimplification. However, a more generalized utility function

- would lead to a more complicated formulation and would not affect the basic findings of this paper.
5. During this period, three companies in the S&P 100 Index were acquired and so were not included in our analysis.
  6. With this transformation, the securities' expected active returns have a correlation with the securities' average daily active returns that is equal to the investor's skill level, and their magnitudes are equal to the magnitudes of the securities' average daily active returns scaled linearly by  $s$ . See Grinold and Kahn (1999). More generally, the Cholesky decomposition of a correlation matrix may be used to form sequences of numbers with a given correlation structure. See, for example, Greene (2011).
  7. When the scaling constant is defined in this way, leverage makes more volatile assets less attractive.
  8. For the market model, see Sharpe (1963). In calculating security and benchmark excess returns, we did not include the daily risk-free rate because it was very close to zero for the analysis period.
  9. Note that volatility and leverage tolerances can be greater than 2. As leverage tolerance increases without limit, the optimal enhancement is that determined by a conventional mean-variance utility function represented by Equation 1.
  10. The surface plot shown is based on a random seed for the variable  $\epsilon_i$  in Equation 13. Different random seeds would produce different expected active returns, but the surface plot would retain its general shape.

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