# Portfolio Insurance, Portfolio Theory, Market Simulation, and Risks of Portfolio Leverage

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## KEY FINDINGS

- Portfolio insurance is a positive-feedback trading strategy that reinforces upward and downward market moves, which can destabilize markets. The rebalancing that portfolio theory implies is a negative-feedback strategy that stabilizes markets.
- Security expected returns can be estimated using cross-sectional analysis, and portfolios can be constructed using mean–variance optimization with suitable constraints. The optimality and optimization of long–short portfolios are addressed.
- JLMSim, an asynchronous, discrete-time, dynamic market simulator populated with investors, traders, and securities, can explain the behavior of security prices and find equilibrium expected returns. A relatively small proportion of momentum investors can overwhelm value investors and destabilize markets. Explosive behavior can result when traders do not anchor their bid or offer prices to existing market prices.
- Portfolio theory is extended with a mean–variance-leverage model to account for the unique risks of leverage, where both volatility aversion and leverage aversion are applied to portfolio choice. The optimal portfolio lies within an efficient region and on a three-dimensional efficient surface. The optimal amount of leverage in 130–30-type portfolio strategies is addressed.

## ABSTRACT

Bruce Jacobs, Ken Levy, and Harry Markowitz shared similar interests and did complementary work. This led to collaboration, debate, and building upon each other's ideas and research. They had a prodigious relationship of over 30 years, bridging the gap between theory and practice. Bruce individually, and then with Harry, distinguished between portfolio insurance and portfolio theory. Bruce and Ken estimated security expected returns using cross-sectional analysis, and Harry used that methodology for portfolio management. Bruce and Ken used Harry's methods for portfolio construction, and they jointly explored the value of using constraints in portfolio optimization and addressed the optimality and optimization of long–short portfolios. Bruce, Ken, and Harry jointly developed an asynchronous, discrete-time, dynamic market simulator, JLMSim, to explain the behavior of security prices and to find equilibrium expected returns. Bruce and Ken extended portfolio theory to account for the unique risks of leverage and applied investor volatility aversion and leverage aversion to portfolio choice. The optimal portfolio lies within an efficient region and on a three-dimensional efficient surface. Harry concurred that the mean–variance model is a special case of the mean–variance-leverage model. Bruce and Ken used the mean–variance-leverage model to address the optimal amount of leverage in 130–30-type portfolio strategies. Bruce and Ken would challenge Harry, and Harry would challenge Bruce and Ken, and out of that would often come something interesting and useful.

Our introduction to Harry Markowitz's groundbreaking work was when we were in graduate school, where his notions of mean–variance optimization, the efficient frontier, and portfolio theory were required reading. Following the founding of our investment firm (Jacobs Levy Equity Management) in 1986, we researched and developed our own insights into security selection and used Harry's methods for portfolio formation.

About a decade later, we were fortunate to work with Harry on some research of mutual interest. We found that one of the most intellectually satisfying experiences was to have a conversation with Harry that led to a question or problem for which none of us had an answer. We would challenge him, he would challenge us, and out of that would come something useful. This essay provides some insight into the way in which our collaboration led to new and interesting ideas.[1] The topic areas are portfolio insurance, portfolio theory, market simulation, and risks of leverage.

## Portfolio Insurance

Soon after we founded our firm (and well before we had completed the research necessary to design our investment process), the market suffered the crash of October 1987. In the years leading up to the crash, portfolio insurance, an option-replication dynamic hedging strategy, was very much in vogue, and Bruce warned that such a positive-feedback strategy had the potential to destabilize markets. He believed that portfolio insurance was a major cause of the crash and wrote a book, *Capital Ideas and Market Realities: Option Replication, Investor Behavior, and Stock Market Crashes* (Jacobs 1999), on the topic. Bruce sent a draft of the book to Harry, who not only liked it but offered to write its foreword (Markowitz 1999). There, in his subtle and piercing way, Harry made the distinction between portfolio insurance and portfolio theory and their differing effects on financial market stability.

## Portfolio Theory

We knew that Harry was not only a portfolio theorist, having literally invented modern portfolio theory, but was also putting his theories into practice, managing a portfolio at Daiwa Securities. We learned that Harry's expected return estimation procedures incorporated our ideas about disentangling the various sources of security price changes. Later, in his foreword (Markowitz 2000) to the first edition of our book, *Equity Management: Quantitative Analysis for Stock Selection* (Jacobs and Levy 2000), Harry referred to our article, "Disentangling Equity Return Regularities: New Insights and Investment Opportunities" (Jacobs and Levy 1988a), as *seminal* and asserted, as we believed, that "such disentangling of multiple equity attributes improves estimates of expected returns."

Stating it somewhat differently and distinguishing signals from noisy stock market data, Charles D'Ambrosio, editor of the *Financial Analysts Journal*, said in 1991:[2]

> They [Jacobs and Levy] were the first to bring so much of this anomaly material together…What they discovered is that there is a lot of noise in the system.

Harry's portfolio management used some of our ideas, and we used his in our portfolio construction. As our relationship with Harry grew, we discovered other common areas of interest. Harry was intrigued by our research examining the conditions under

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[1] See also Jacobs forthcoming, "Collaborating with Harry Markowitz: A Remembrance."
[2] See White (1991).

which an optimal long-short portfolio would be naturally dollar or beta neutral (Jacobs, Levy, and Starer 1998). We were both interested in and conducted joint work on efficient ways to compute optimal portfolios that included short positions. This led to the development of theorems regarding the conditions under which standard efficient algorithms could be applied to the long–short problem (Jacobs, Levy, and Markowitz 2005) and to the concept of "trimability" (Jacobs, Levy, and Markowitz 2006).

### Market Simulation

Harry's intellectual arsenal included his skill in computer simulation. Like us, Harry was a computer nerd, and he often liked to joke that when his doctor asked how he was doing, he would reply: "Not so good. I've got a bug again."

Harry not only created portfolio theory but was a leading figure in the simulation world—he created the language SIMSCRIPT. We were very interested in studying the behavior of financial markets in response to various stimuli, but (for reasons we will discuss later) models that could simulate realistic markets were not available. Thus, we teamed up with Harry to create the Jacobs Levy Markowitz Market Simulator (JLMSim), as described in Jacobs, Levy, and Markowitz (2004, 2010).

### Risks of Portfolio Leverage

We presented to Harry an extension of portfolio theory to account for the unique risks of leverage and had a friendly debate (Jacobs and Levy 2013a, 2013b; Markowitz 2013). We explored the impact of leverage on portfolio risk, highlighting how it can lead to margin calls that force borrowers to liquidate securities at adverse prices. Our mean-variance-leverage model finds optimal portfolios with the right amount of leverage and the right kind of diversification, taking into account both investors' volatility aversion and their aversion to the unique risks of leverage.

## PORTFOLIO INSURANCE

In Harry's view, the causes of the crash of October 19, 1987, should be studied so that one can understand a tumultuous event in stock market history and also so that one can grasp the implications for stock market mechanisms and their possible consequences. Harry believed, as Bruce did, that the severity of the 1987 crash was due, in large part, to the use of an option replication strategy known as portfolio insurance.

Bruce had been quite vocal throughout the 1980s about the dangers of portfolio insurance, beginning when he was asked to assess the strategy while he was working for the Prudential Insurance Company of America. In a January 17, 1983, memorandum to Prudential executives (Jacobs 1999, Appendix A; Jacobs 2018, Appendix C), he warned that the dynamic hedging strategy's automatic, trend-following trading could destabilize markets and cause the insurance to fail. Prudential heeded the warning and decided not to offer the ill-fated strategy.

Bruce wrote articles revealing the real costs of the strategy and engaged in a series of public debates with the principals of Leland O'Brien Rubinstein Associates, the firm that had created portfolio insurance based on the Black–Scholes–Merton option pricing formula (see Jacobs 1999 and 2018). The *Wall Street Journal* (Anders 1986) and *Pensions & Investment Age* (Jacobs 1987) recognized Bruce as being among the first and most prominent to sound the alarm about portfolio insurance.

A more formal exploration of the role of portfolio insurance in destabilizing markets began as a manuscript, "The Rise of Portfolio Insurance and the Crash of 1987." Bruce began circulating it to colleagues in 1990. He was intrigued by the response of Paul Samuelson (private letter to Bruce dated July 2, 1990), who characterized portfolio insurance as a source of reassurance for traders who saw stock valuations as stretched in 1987 but believed (wrongly, as it turned out) that they could "make a fast exit after the turn, beating most of the mob."

Harry's response to the manuscript was brief but encouraging (letter dated July 3, 1990): "I find it comprehensive, scholarly and convincing." Several years later, Bruce sent him a substantially updated manuscript, and he replied with a lengthy and thoughtful letter (dated March 19, 1997). Emboldened, Bruce called Harry. Toward the end of their extensive conversation, Harry said, "Bruce, is there something you wanted to ask me?" Bruce replied, "Yes, Harry, in fact there is. Would you provide a foreword for the book?" Harry replied, "I would be delighted to. Of course, it would depend upon whether I find something interesting to say."

Of note, within weeks of the Nobel Prize being awarded for the option pricing model on October 14, 1997, Roger Lowenstein wrote a *Wall Street Journal* article (November 6, 1997), "Why Stock Options are Really Dynamite," which referred to Bruce's just finished manuscript now called "Capital Ideas and Market Realities," and said: "So what do option-writers—that is, people who provide insurance—do? Many employ a strategy known as 'dynamic hedging.' In a nutshell, they try to sell stocks on the way down—enhancing the trend and at once making the strategy futile for the group. This is the same failed tactic of a decade ago," which was a reference to the failure and role of portfolio insurance in the Crash of 1987.

In the foreword to Bruce's 1999 book *Capital Ideas and Market Realities: Option Replication, Investor Behavior, and Stock Market Crashes*, Harry (Markowitz 1999) proposed a very simple thought experiment to compare the mean–variance efficiency of trading based on portfolio theory with that based on portfolio insurance.

Consider portfolios consisting only of a single security (the "market") and cash. Because portfolio insurance does not make use of beliefs about market movements, one can assume that the market's returns are independent and identically distributed. For simplicity, assume that the portfolio can be switched back and forth between cash and the market without cost.

To get some idea of the performance of portfolio insurance, compare a simple version of it with a constantly rebalanced portfolio. The portfolio insurance rule could be any function of past observations but, to obtain specific results, assume that over some number of time periods, the portfolio insurance rule directs the investor to be completely in cash for half the time and completely in the market for the other half. The alternative strategy is to rebalance the portfolio at each period to be half in the market and half in cash. Now compute the realized mean return and the variance of return for these strategies. Because trades executed by portfolio insurance strategies are not motivated by shifting beliefs about the market's movements, one can assume that such strategies' performance per period is a sequence of random draws.

With these simple assumptions, Harry computed the means and variances of the returns of the "switch back and forth" (i.e., portfolio insurance) strategy and the rebalanced strategy. The expressions are given in Exhibit 1, in which $\mathbb{E}[r]$ is the mean of the strategies' return, $\mathbb{V}[r]$ is the variance of the returns, $r_m$ is the mean value of the market's return, $\sigma_m^2$ is the variance of the market's return, and $r_0$ is the return on cash.

Exhibit 1 shows that switching back and forth between cash and stocks is detrimental to mean–variance efficiency, *even assuming zero transaction costs.* These two strategies have the same mean return. However, the strategy of switching back and forth has more than twice the return variance of the strategy of rebalancing to a 50/50 portfolio. The rebalanced strategy will be on the market line, whereas the strategy of switching back and forth will be below the market line. This holds true for whatever the proportions of stock and cash chosen, given the assumption of independent and identically distributed returns. Even if one uses semivariance as a measure of risk, as discussed in Markowitz (1959, Ch. 2), the rebalanced portfolio

## EXHIBIT 1

### Mean and Variance of Return for Two Trading Strategies

| | $\mathbb{E}[r]$ | $\mathbb{V}[r]$ |
|---|---|---|
| Rebalanced 50/50 portfolio | $(r_m + r_0)/2$ | $\sigma_m^2/4$ |
| Switch back and forth | $(r_m + r_0)/2$ | $\sigma_m^2/2 + (r_m - r_0)^2/4$ |

will still be on the (semivariance) market line, whereas the strategy that switches back and forth will be below the line.

The example provided is, of course, an extreme case in which the portfolio insurer is either in stocks or in cash, but not both simultaneously. The direction of the result is the same, however, in the more realistic, less extreme, case where the proportion of stocks held by the portfolio insurer varies over time as a function of past observations. With his characteristic wit, Harry conceived a fictitious debate in which a portfolio *insurance* supporter sparred with a portfolio *theory* supporter over the example.

In the debate, the portfolio insurance supporter argued that the strategy's mean–variance inefficiency is the price paid to reshape the probability distribution of returns over a longer interval of time, that is, the insured period. The portfolio insurance supporter pointed out that, under reasonable assumptions, the greatest loss in any period is less for the portfolio insurance strategy than for the rebalanced strategy and, given the assumptions, in no month does the loss exceed the preset floor.

In response, the portfolio theory supporter countered that, for the period of analysis as a whole, the rebalanced portfolio grew more than the switched-back-and-forth one. This is because the strategy with the greater average logarithmic return[3] will have grown the most during the period; average logarithmic return is very closely approximated by a function of return mean and variance (see Markowitz 1959, Ch. 6); and this approximate average decreases with increasing variance. A particular point on the mean–variance frontier gives approximately maximum growth. Moreover, every point on the frontier gives approximately maximum growth in the long run for given short-run fluctuations.

In rebuttal, the portfolio insurance supporter countered that, after one or two bad years, an investor (the client) hiring an investment manager (the agent) using the rebalancing strategy might not wait for the long run but might summarily fire the investment manager. To which, the portfolio theory supporter might contrast the needs of the client with those of the agent, thereby seeking the moral high ground. The portfolio insurance supporter might state that, in practice, applications of portfolio theory sometimes put investment manager motives ahead of true client needs, as perhaps when mean–variance analysis is used for a given average return to minimize tracking error rather than total variability. Finally, the portfolio theory supporter could deliver the winning argument by stating that, whatever the arguments pro and con, the debate was irrelevant because portfolio insurance simply did not work in practice, especially when it was needed the most.

It is still instructive, however, to study the effects of portfolio insurance. It destabilized the market, creating liquidity problems that effectively caused it to fail. Those following portfolio insurance strategies bought stocks when the market went up and sold stocks when the market went down. Such a strategy creates positive feedback, reinforcing upward market moves and exacerbating downturns. Destabilization is a well-known consequence of positive feedback. In contrast, the rebalancing strategies that portfolio theory implies call for selling stocks when the market rises and buying

[3] By logarithmic return, we mean $\log((p_{t+1} + d_t)/p_t)$, where $p_t$ is the price at time $t$ and $d_t$ is the dividend paid over the period $t$ to $t + 1$.

stocks when the market falls.[4] This is negative feedback, and it tends to stabilize the market. Thus, in Harry's words, "such an application of portfolio theory is, if nothing else, more environmentally friendly [to the market] than portfolio insurance."

Jacobs (2004, 2009) examined more recent episodes of market instability attributable to positive feedback strategies. In particular, in the middle of the decade of the aughts, structured finance instruments, including residential mortgage-backed securities and collateralized debt obligations, as well as a type of credit insurance known as credit default swaps, facilitated a positive-feedback system. The ability to transfer risk from lenders to investors and insurers encouraged mortgage lending and lowered mortgage rates, enabling a housing bubble to develop. Increasing home prices in turn increased demand for mortgages and creation of more structured finance products and credit default swaps. As the experience with portfolio insurance should have taught us, however, shifting risk may change *who* the risk holders are, but it does not reduce overall risk and can actually increase it. When housing prices leveled out and started to decline in 2006, the same products that had been used to (supposedly) reduce risk ended up spreading risk through the entire financial system, leading to the Global Financial Crisis of 2008–2009.

Instruments and strategies that purport to reduce systematic risk for portfolios, including stock and mortgage portfolios, can end up increasing risk for the overall financial system. Risk bearers need to be able to withstand unexpected losses; otherwise, the risk can become systemic and, as Jacobs (2004) warned, fall on taxpayers. In 2008–2009, the risk bearers, including financial institutions that bought structured products and underwrote credit default swaps, failed. The government—or rather, the taxpayers—had to step in as the risk bearer of last resort.

## PORTFOLIO THEORY

In Harry's foreword (Markowitz 2000) to our book, *Equity Management: Quantitative Analysis for Stock Selection*, he wrote, "It may be fairly asserted that Jacobs and Levy's work is based on mine, and my work is based on theirs." He points out that we, as do almost all quantitative investment firms, make use of the general mean–variance portfolio selection model presented by Markowitz (1956, 1959), which in turn extended a proposal in Markowitz (1952). This is the sense in which some of our work is based on his.

To be practically applicable, mean–variance analysis as presented in Markowitz (1952, 1956, 1959) requires estimates of the means and variances of the returns of individual securities, as well as covariances between returns of pairs of securities. But those pioneering articles did not specify how to make these estimates. In fact, Markowitz (1952) begins:

> The process of selecting a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage.

It turned out that when Harry addressed the first stage and turned his hand to portfolio management, he and his colleagues used expected return estimation procedures based on Jacobs and Levy (1988a), as cited in Bloch et al. (1993).

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[4] Investors often establish a policy portfolio consisting of an unleveraged mix of equity, fixed income, and other assets. If equities rise in price, the percentage of the portfolio held in equities would exceed the policy portfolio's percentage, and equities would be sold (absent a change in the investor's beliefs) to rebalance the portfolio.

We devised a multivariate approach to return estimation that took into account a multitude of factors and their interrelationships (Jacobs and Levy 1988a), which Harry described in his own words:

> Before 1988 anomaly studies considered small numbers of variables, usually one to three at a time. Observing that some apparent anomalies may be surrogates for others, Jacobs and Levy fit a series of monthly cross-sectional regressions of security excess returns against 25 anomaly and 38 industry variables. This allowed them to "disentangle" what they called the "pure" (i.e., underlying) anomalous effects from what they called the "naïve" effects observed from simple regressions against anomalous variables one at a time. The Jacobs and Levy methodology may be used for expected return estimation as well as for explaining observed anomalies. Further work along these lines includes Haugen and Baker (1996) and Schwartz and Ziemba (2000). (Markowitz and Van Dijk 2006)

The cross-sectional analysis that we pioneered has greater explanatory power than the time-series approach based on portfolio sorts (such as the return differences between small- and big-capitalization stocks and between high- and low-book-to-price stocks) that has dominated the asset pricing literature (Jacobs and Levy 2021).

We also examined how abnormal equity returns were associated with the turn of the year, the week, and the month, as well as with holidays and time of day, and how payoffs to the size effect may be predictable using such macroeconomic drivers as interest rates and industrial production (Jacobs and Levy 1988b, 1989).

We'll now describe some of our portfolio theory research, research that we have done jointly with Harry, and work we have done building on Harry's portfolio theory.

## Integrated Long–Short Optimization and Portfolio Constraints

In order to optimize a portfolio, one should not impose any unnecessary constraints. In Jacobs, Levy, and Starer (1998), we defined a minimally constrained portfolio that maximizes expected investor utility and argued that imposing any other constraints can reduce utility. In that paper, we also defined and advocated the use of "integrated optimization."

In the foreword to our *Equity Management* book (Markowitz 2000), Harry noted that our work on integrated optimization of long–short portfolios and the estimation of security expected returns was "to be acknowledged for bridging the gap between theory and practice in the world of money management." He went on to say that the translation of investment ideas into products and strategies must involve trade-offs between theory and practice. He then discussed why, in the portfolio optimization problem, investors might want to add constraints on position sizes and sectors, despite the theoretical cost of these constraints.

As Harry explained with reference to Markowitz (1959, Ch. 13), the mean–variance investor approximates a rational decision maker (RDM) acting under uncertainty. The mean–variance optimal portfolio may be less averse to an extreme downside move than the one that optimizes an investor's true (i.e., subjective) expected utility (see Table 1 in Haim Levy and Markowitz 1979). In Harry's words: "It is therefore possible that adding constraints to a minimally constrained mean–variance analysis may produce a portfolio that gives higher true expected utility, even though it gives a lower value to a mean–variance approximation" (Markowitz 2000).

Nevertheless, as we pointed out in Jacobs, Levy, and Starer (1998), a general principle of optimization is that constrained solutions do not offer the same level of utility as unconstrained solutions unless, by some fortunate coincidence, the optimum lies within the feasible region dictated by the constraints.

Treynor and Black (1973, p. 66) had hinted at similar issues and specifically posed the following question: "Where practical is it desirable to so balance a portfolio between long positions in securities considered underpriced and short positions in securities considered overpriced that market risk is completely eliminated?" We reformulated Treynor and Black's question slightly, posing the following three questions:

1. Under what conditions will a net holding of zero (i.e., dollar-neutrality) be optimal for a long–short portfolio?
2. Under what conditions will the combined optimal holdings in a long–short portfolio be beta-neutral?
3. Under what conditions will dollar-neutrality or beta-neutrality be optimal for the active portion of an equitized long–short portfolio?

To answer these questions, consider the standard mean–variance utility function:

$$U = E_P - \frac{1}{2\tau} V_P, \tag{1}$$

where $E_P$ is the expected return on the investor's portfolio, $V_P = \sigma_P^2$ is the variance of the return and $\tau$ is the investor's risk tolerance.

Assume that in seeking to maximize the utility function in (1), the investor has an available capital of $K$ dollars and has acquired $n_i$ shares of security $i \in \{1, 2, \ldots, N\}$. A long holding is represented by a positive number of shares, and a short holding is represented by a negative number. The holding $h_i$ in security $i$ is the ratio of the amount invested in that security to the investor's total capital. Thus, if security $i$ has price $p_i$, then $h_i = n_i p_i / K$. With these definitions, the portfolio's return has mean and variance given by

$$E_P = h^\top r \tag{2}$$

$$\sigma_P^2 = h^\top Q h, \tag{3}$$

where $h$ is a vector containing the individual holdings, $r$ is the vector of expected security returns, and $Q$ is the covariance matrix of the securities' returns.[5] Using (2) and (3), one finds that the *unconstrained* portfolio vector that maximizes the utility in (1) is

$$h = \tau\, Q^{-1} r. \tag{4}$$

This unconstrained portfolio will be naturally dollar neutral (i.e., dollar neutral without the need to impose any constraints) if the net holding, $H$, is zero. Using a constant correlation model as described in Elton, Gruber, and Padberg (1976), we found that this net holding is closely approximated by

$$H = \frac{\tau}{1-\rho} \sum_{i=1}^{N} (\xi_i - \bar{\xi}) \frac{r_i}{\sigma_i}, \tag{5}$$

where $\rho$ is the correlation from the constant correlation model, and for security $i$, $r_i$ is the expected return, $\sigma_i$ is the standard deviation of the return, $\xi_i = 1/\sigma_i$ is a measure of the return stability, and $\bar{\xi}$ is the average of all the $\xi_i$.

The net holding in (5) will be zero either in the trivial case when the risk tolerance is zero or in the more interesting case when the sum is zero. The sum can be regarded

[5] Similar expressions are obtained whether one works with absolute or excess returns.

as the net risk-adjusted return ($r_i/\sigma_i$) of all securities weighted by the deviation ($\xi_i - \bar{\xi}$) of their stability from the average stability.[6] If this sum is positive, the net holding should be long. Conversely, if this sum is negative, the net holding should be short.

The fractional change in utility when dollar neutrality is imposed is

$$\frac{\Delta U}{U} = -\frac{(\mathbf{1} + Q^{-1}r)^2}{(\mathbf{1}^{\top} Q^{-1}\mathbf{1})(r^{\top} Q^{-1} r)},$$

where **1** is a vector of ones. This change has a maximum value of zero (which occurs when the condition for dollar neutrality is satisfied) and is otherwise always negative. Thus, only under the special condition in which $H$ in (5) is equal to zero will the optimal portfolio be dollar neutral. Constraining the holding to be zero when this condition is not satisfied will produce a suboptimal portfolio; that is, one with decreased (mean–variance) utility.

To answer the second question about beta-neutrality, we performed a similar analysis, using Sharpe's single index model. In this case we found that the beta of an unconstrained portfolio is approximately,

$$\beta_P = \frac{\tau}{1-\rho}\sum_{i=1}^{N}(\beta_i - \bar{\beta})\frac{r_i}{\sigma_i}, \tag{6}$$

where

$$\bar{\beta} = \frac{1}{N}\sum_{i=1}^{N}\frac{\beta_i}{\sigma_i}.$$

Equation (6) is entirely analogous to (5): The sum can be regarded as the net risk-adjusted return ($r_i/\sigma_i$) of all securities weighted by the deviation ($\beta_i - \bar{\beta}$) of their beta from the volatility-weighted beta. The net beta should have the same sign as this sum. Only under the special condition in which $\beta_P$ in (6) is equal to zero will the optimal portfolio be beta neutral. Constraining the beta to be zero when this condition is not satisfied will produce a suboptimal portfolio. Thus we concluded that only under very specific conditions will dollar or beta neutrality be optimal.

The same conclusions hold with regard to the third question. That is, only under very specific conditions will an equitized long–short portfolio hold long and short positions that are balanced by dollar or beta. Furthermore, the degree of equitization itself becomes a matter of optimization. As we stated in Jacobs, Levy, and Starer (1998, p. 40), "The important question is not how one should allocate capital between a long-only portfolio and a long–short portfolio but, rather, how one should blend active positions (long and short) with a benchmark security in an integrated optimization." Jacobs, Levy, and Starer (1999) showed that a theoretically optimal portfolio would be constructed in a single, integrated optimization that considers the expected returns, risks, and correlations of all securities, including any benchmark, simultaneously. Such a portfolio will rarely be naturally totally neutral with respect to any particular characteristic.

Of course, there may be perfectly valid tax, accounting, or regulatory reasons for dollar-neutral, beta-neutral, market-neutral, or fully equitized portfolios. Such portfolios may also be preferred for behavioral reasons, such as mental accounting, or because they fit more readily into established frameworks for performance evaluation and comparison. But perhaps such constrained portfolios merely reflect,

[6] Note that the sum is not a weighted average because the weights do not sum to 100%, and some weights are in fact negative.

as Harry argued, "the inability of human decision makers to fully emulate RDMs [Rational Decision Makers] in maximizing expected utility in the face of uncertainty and illiquidity" (Markowitz 2000).

The general theoretical conclusion, however, is that imposing neutrality moves the portfolio away from mean–variance optimality. The corollary to this finding is that determining equity market exposure should be done as part of determining individual security positions: active long and short positions, as well as benchmark holdings, should be determined jointly, in an integrated optimization. Harry's 1952 tenet still holds: Mean–variance analysis provides the right kind of diversification for the right reason. However, imposing unnecessary constraints can render a portfolio mean–variance suboptimal.

### Trimablity of Long-Short Portfolios

Following our work on the optimality of long–short portfolios and the benefits of integrated optimization, we turned our attention to fast methods for optimizing such long–short portfolios subject to realistic constraints. Harry's expertise in optimization theory proved invaluable: Optimizing quadratic functions subject to linear constraints (see Markowitz 1956) is applicable to the optimization of long–short portfolios.

The portfolio optimization is a quadratic function optimization as follows: Consider a portfolio consisting of $n$ securities with expected returns $\mu_1, \mu_2, \ldots, \mu_n$. The portfolio can include both risky and riskless securities. The portfolio's expected return, $E_P$, is a weighted sum of the $n$ security returns:

$$E_P = \sum_{i=1}^{n} x_i \mu_i, \tag{7}$$

where $x_1, x_2, \ldots, x_n$ are the security weights in the portfolio.[7] If the covariance between the returns of security $i$ and security $j$ is $\sigma_{ij}$, the portfolio's return variance, $V_P$ is

$$V_P = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \sigma_{ij} x_j. \tag{8}$$

The security weights may be subject to various constraints. For long-only portfolios, common constraints include the following:

$$\sum_{k=1}^{n} a_{jk} x_k = b_j, \quad \text{for } j = 1, \ldots, m \tag{9}$$

and

$$x_i \geq 0, \quad \text{for } i = 1, \ldots, n, \tag{10}$$

where $m$ is the number of constraints. Equation (9) might include, for example, a budget constraint according to which the sum of the weights must equal a fixed number. Equation (10) is a nonnegativity constraint.

The general single-period mean–variance portfolio selection problem is, for all variances, $V_P$, find the corresponding portfolios that provide maximum expected return $E_P$, or alternatively, for all expected returns $E_P$, find the corresponding portfolios that provide minimum variance $V_P$, subject to the given constraints.

[7] Note that the notation used in each section of this article is consistent with the original articles.

For a long–short portfolio, the sign of $x_i$ is not constrained. A negative value of $x_i$ is interpreted as a short position. Unfortunately, with such an interpretation, unrealistic portfolios can be obtained. For example, if the portfolio is subject only to the full investment constraint, as in the capital asset pricing model (CAPM), an investor could deposit $1,000 with a broker, short $1,000,000 of Stock A, and use the proceeds plus the original deposit to purchase $1,001,000 of Stock B. Short positions do not work this way.

Although no single constraint set applies to all long–short portfolios, all constraints of practical interest can be accommodated if one adopts the convention of representing an $n$-security long–short portfolio in terms of $2n$ nonnegative variables, $x_1, \ldots, x_{2n}$, in which the first $n$ variables represent the securities in a given set held long, the second $n$ variables represent short positions in the same set of securities, and one chooses the long–short portfolio subject to the following constraints:[8]

$$\sum_{k=1}^{2n} a_{jk} x_k = b_j, \quad \text{for } j = 1, \ldots, m \tag{11}$$

and

$$x_i \geq 0, \quad \text{for } i = 1, \ldots, 2n. \tag{12}$$

The types of constraints incorporated in constraints (11) and (12) include budget constraints, upper and lower bounds on long and short positions, equality constraints on particular positions, market-neutrality constraints, constraints on net long or short positions, or on borrowing or margins. An apparent disadvantage of (11) and (12), insofar as portfolio optimization is concerned, is that they allow long and short positions in the same security. We consider this issue later.

In general, the covariances, $\sigma_{ij}$, in (8) are nonzero, so the covariance matrix will be dense (i.e., will contain mostly nonzero entries). The solution of the general mean–variance portfolio selection problem requires the inversion of a matrix that includes this covariance matrix as one of its blocks. This inversion is one of the major computational burdens in portfolio optimization.

It was unclear whether fast portfolio optimization algorithms, which were applicable to long-only portfolios, were applicable to long–short portfolios as well. Long–short portfolios can take many forms, including market-neutral equity portfolios that have a zero market exposure and enhanced active equity portfolios that have a full market exposure, such as 130–30 portfolios (with 130% of capital long and 30% short). While studying the problem of optimizing long–short portfolios with Harry, we collectively came up with the notion of "trimability." This is a sufficient condition under which a fast portfolio optimization algorithm designed for long-only portfolios will find the correct long–short portfolio, even if the algorithm's use would violate certain assumptions made in the formulation of the long-only problem.[9] In the following, we briefly describe the basic approach of using covariance models to design fast portfolio optimization algorithms and then discuss the trimability condition under which such algorithms are also applicable to long–short portfolios.

For long-only portfolios, there are at least three types of models—factor models, scenario models, and historical models—that can be used to transform the portfolio selection problem into one that requires the inversion of a diagonal (or nearly diagonal) matrix.

[8] Because the second $n$ variables represent short positions in the same set of securities, if security $i$ is held long, $x_i$ will be positive, and if security $i$ is sold short, $x_{n+i}$ will be positive.

[9] The mathematical specifics of this condition are described in detail in Jacobs, Levy, and Markowitz (2005).

Diagonal matrices are easy to invert, so their use in place of denser matrices can greatly simplify and speed the optimization problem. The "trick" to obtaining diagonal matrices for long-only portfolios is to introduce fictitious securities that are linearly related to the original securities but constrained in some way. For example, consider a factor model in which $r_i$, the return of security $i$, is given by

$$r_i = \alpha_i + \sum_{k=1}^{K} \beta_{ik} f_k + u_i, \quad \text{for } i = 1, \ldots, n, \tag{13}$$

where $\alpha_i$ is a constant, $f_k$ is the return on the $k$-th common factor, $\beta_{ik}$ is the factor loading, $K$ is the number of common factors, and $u_i$ is an idiosyncratic term assumed to be uncorrelated with $u_j$ for all $i \neq j$ and uncorrelated with all $f_k$ for $k = 1, \ldots, K$. For simplicity, we also assume that $f_k$ is uncorrelated with $f_j$ for $j \neq k$.[10]

To perform the diagonalization, one introduces fictitious securities, one for each common factor (see Sharpe 1963; Cohen and Pogue 1967), with the weight of each fictitious security constrained to be a linear combination of the weights of the real securities. Accordingly, one defines a set of $K$ fictitious securities with weights $y_1, \ldots, y_K$ in terms of the real securities as follows:

$$y_k = \sum_{j=1}^{n} x_j \beta_{jk}, \quad \text{for } k = 1, \ldots, K. \tag{14}$$

With this definition, the portfolio variance can be written (see Jacobs, Levy, and Markowitz 2005) in the form

$$V_P = \sum_{i=1}^{n} x_i^2 V_i + \sum_{k=1}^{K} y_k^2 W_k, \tag{15}$$

where $W_k$ is the variance of $f_k$. Equation (15) expresses $V_P$ as a positively weighted sum of squares in the $n$ original securities and $K$ new fictitious securities, which are linearly related to the original securities by (14).

Note that the variance expression in (15) contains only two single sums (whereas the variance expression in (8) contained a nested double sum). Therefore, (15) can be written in terms of a diagonal covariance matrix; that is, we have effectively diagonalized the model.

We showed in Jacobs, Levy, and Markowitz (2005, 2006) that analogous procedures can be used to write scenario models and historical models in diagonal form. We refer to these models as "diagonalizable models." In each case, diagonalization transforms the variance expressions from ones couched in terms of dense covariance matrices to ones containing matrices that are slightly larger but have nonzero entries only along their diagonals. Inversion of such matrices is trivial.

Can this diagonalization procedure, used for long-only portfolios, be applied to the optimization of long–short portfolios? To investigate this question, we adopt the convention of representing an $n$-security long–short portfolio in terms of $2n$ nonnegative variables $x_1, \ldots, x_{2n}$. Let $r_c$ be the return on cash or collateral. The portfolio's return $R_P$ is then

$$R_P = \sum_{i=1}^{n} r_i x_i + \sum_{i=n+1}^{2n} (-r_{i-n}) x_i + r_c \sum_{i=n+1}^{2n} h_{i-n} x_i. \tag{16}$$

[10] The mathematical details of the more general case, in which the factors are not necessarily mutually uncorrelated, are discussed in Jacobs, Levy, and Markowitz (2005).

The first term on the right-hand side of (16) represents the return contribution of the securities held long. The second term represents the contribution of the securities sold short. The third term represents the short rebate, where

$$h_i \le 1, \quad \text{for } i = 1, \ldots, n$$

is the investor's portion of the interest received on the proceeds from the short sale of security $i$. With these definitions, the returns on the short positions are

$$r_i = h_{i-n} r_c - r_{i-n}, \quad \text{for } i = n+1, \ldots, 2n. \tag{17}$$

Let $\mu_i$ be the expected value of $r_i$, for $i = 1, \ldots, 2n$. Then, the expected return of the long–short portfolio is

$$E_P = \mathbb{E}[R_P] = \sum_{i=1}^{2n} x_i \mu_i. \tag{18}$$

To diagonalize, we assume a multifactor model with returns given by (13) and we define $K$ new fictitious securities, $y_1, \ldots, y_K$, in terms of the real securities, as follows:

$$y_k = \sum_{j=1}^{n} x_j \beta_{jk} - \sum_{j=1}^{n} x_{n+j} \beta_{jk}, \quad \text{for } k = 1, \ldots, K.$$

From this definition, it follows (see Jacobs, Levy, and Markowitz 2005) that the variance of the portfolio's return is

$$V_P = \sum_{i=1}^{2n} x_i^2 V_i + \sum_{k=1}^{K} y_k^2 W_k - 2 \sum_{i=1}^{n} x_i x_{n+i} V_i. \tag{19}$$

Equation (19) is the expression for the variance of the return of a long–short portfolio when a multifactor covariance model is assumed. Note that, with the exception of the cross-product terms, (19) has exactly the same form as (15). Had the cross-product terms $x_i x_{n+i}$ been zero, the model for the long–short portfolio would have been diagonal.

Recall that $x_i$ is the magnitude of a long position in security $i$ and $x_{n+i}$ is the magnitude of a short position in security $i$. Therefore, if the cross-products are all zero, the portfolio has no simultaneous long and short positions in the same securities because either $x_i$ or $x_{n+i}$ is zero or both are zero. We refer to such a long–short portfolio as a "trim" portfolio. Mathematically, a trim portfolio has

$$x_i x_{n+i} = 0, \quad \text{for } i = 1, \ldots, n.$$

Trim portfolios have the useful property that, for them, (19) has precisely the same form as (15); that is, their covariance matrices (including fictitious securities) are diagonal.

Conceptually,[11] if, using the given return model, we are able to transform a feasible portfolio that is untrim (i.e., one that has at least one security in which it has simultaneous long and short positions) into a feasible portfolio that is trim in a way that does not reduce the portfolio's expected return, the model satisfies the

[11] More precise statements are provided in Jacobs, Levy, and Markowitz (2005, 2006).

"trimability condition."[12] Such a model is called "trimable." Importantly, we can apply existing fast portfolio optimization algorithms to trimable long–short portfolio models.

For a guarantee that an efficient set for a model in which the cross-product terms are ignored is an efficient set for the model in which they are not, we must be able to trim the model in the following way:

- remove the overlap from simultaneous long and short positions in each security in such a way that the smaller of the two positions diminishes to zero,
- add the overlap to a risk-free security holding,
- leave all other risky security holdings unchanged,
- maintain feasibility, and
- not reduce the expected return of the portfolio.

Although models with arbitrary constraint sets may not satisfy the trimability condition, a wide variety of constraints met in practice do satisfy it. In Jacobs, Levy, and Markowitz (2005, 2006), we provide examples of models that can be trimmed, as well as examples that cannot. We also provide tables that show the dramatic improvement in computational speed that can be achieved using fast algorithms to optimize trimable long–short portfolios.

### Trim Equitized and Enhanced Active Equity Equivalence

Jacobs and Levy (2007) applied the concept of trimability to illustrate the relationship between equitized market-neutral long–short (ELS) portfolios and enhanced active equity (EAE) portfolios, such as 130–30 portfolios, and to show specifically that every ELS portfolio has an equivalent EAE portfolio, and vice versa.

In EAE portfolios, the strict long-only constraint is relaxed so that the manager can sell stocks short up to some prespecified percentage of capital (e.g., 30%), and use the proceeds of the short sales to buy additional long positions. The overall portfolio thus has 130% of its capital long and 30% short. Overall, it maintains a 100% exposure to the market.

An ELS portfolio also has a 100% exposure to the market, achieved with stock index futures or exchange-traded funds (ETFs), and it has a long–short component that may have 100% of capital long and 100% of capital short.

The EAE portfolio is essentially a compact form of the ELS portfolio. If the ELS portfolio contains short positions in stocks that are held in the equitizing instrument (i.e., in the underlying index of the stock index future, or in the ETF), then the ELS portfolio is untrim. While the ELS portfolio may not be trimable in practice because individual securities in the equitizing instrument cannot be sold to remove overlaps, there is a unique EAE portfolio that is functionally identical to, but more compact than, the untrimmed ELS portfolio.[13]

Consider a market-neutral long–short portfolio that has $100M$% of capital long and $100M$% short, where $M$ is a multiple of the investor's capital.[14] An equitized portfolio consisting of this market-neutral long–short portfolio and a benchmark index overlay is equivalent to an enhanced active equity portfolio with $100(1 + E)$% held long and $100E$% sold short. Here, $E$ is a quantity that we call the enhancement, equal to

$$E = M - T,$$

[12] This condition is called "Property P" in Jacobs, Levy, and Markowitz (2005).

[13] For transaction cost differences between EAE and ELS portfolios, see Jacobs and Levy (2007).

[14] When $M = 1$, the portfolio is a fully invested market-neutral long–short portfolio with 100% of capital long and 100% of capital short.

where

$$T = \sum_{i \in S} \min\{|x_i|, b_i\}$$

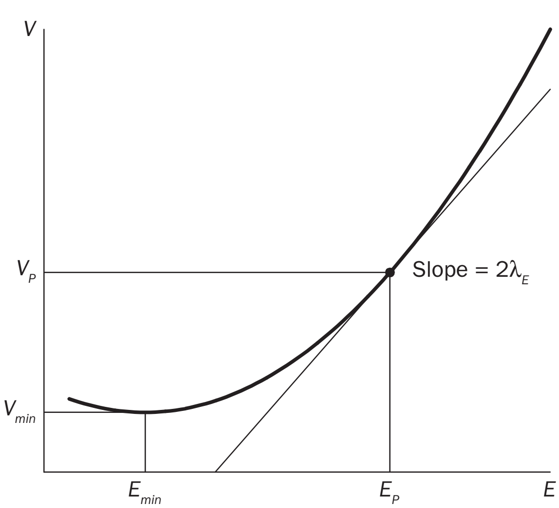
is the fraction of capital trimmed to eliminate simultaneous long and short exposures to the same security, $x_i$ is the weight of the $i$-th security in the market-neutral long–short portfolio, $b_i$ is its weight in the benchmark, and $S$ is the set of securities sold short in the market-neutral long–short portfolio. The trimmed amount, $T$, has a minimum value of zero (corresponding to the case where there is no overlap) and a maximum value of 1 (corresponding to the case where there is complete overlap). More details, including a comparison of EAE portfolios with ELS portfolios, and examples of equivalent portfolios, are provided in Jacobs and Levy (2007).

### Efficient Frontier Algorithmic Equivalence

In considering fast algorithms, we found that some algorithms appeared to take completely different approaches yet produce the same efficient frontier. In particular, under realistic assumptions, there is a piecewise linear set of portfolios that supplies one and only one efficient portfolio for each efficient risk–return combination. If the covariance matrix is nonsingular, then this set of efficient portfolios is unique. Thus, any algorithm that traces out the mean–variance efficient frontier must produce the same result. One such algorithm is the critical line algorithm (CLA; described in detail by Markowitz 1956, 1959, 1987), which is an iterative technique, applicable to the general portfolio problem. Sharpe (1963) presented a procedure that greatly simplifies the CLA computation, specifically for a one-factor model of covariance with long positions only. Elton, Gruber, and Padberg (1978) presented an alternative algorithm (EGP) for finding the efficient frontier for various special models, including the one-factor model with long positions only. Though they must produce the same efficient frontier, the two algorithms are parameterized differently.

Both the CLA and the EGP algorithm trace out the same efficient frontier by varying a parameter in discrete steps over a certain range and finding the corner portfolio that corresponds to the value of the parameter at each step. The CLA is applicable to arbitrary covariance models, while the EGP algorithm applies only to certain specific models of covariance. The uniqueness of the efficient frontier guarantees that the two algorithms must both find the same set of corner portfolios. Therefore, there must exist a unique relationship between the parameters used in the algorithms. In Jacobs et al. (2007), we explained that relationship for long-only portfolios with the assumption that the investor can neither lend nor borrow at the risk-free rate.

**EXHIBIT 2**
**Geometry of the CLA**

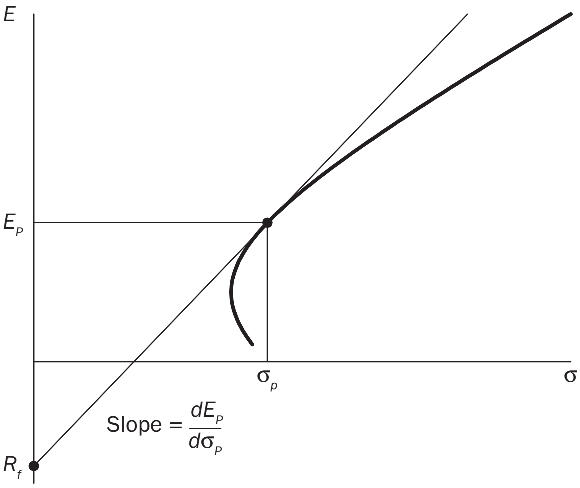



The relationship between $E_P$ and $V_P$ along the efficient frontier is shown in Exhibit 2. This exhibit draws expected return on the horizontal axis, and variance on the vertical axis, as done in Markowitz (1987). This differs from the current convention of drawing standard deviation on the horizontal axis and expected return on the vertical axis. The curved line in the exhibit represents the efficient frontier itself. The parameter $\lambda_E$ can be interpreted as half the slope of the efficient frontier in ($E_P$, $V_P$) space; or because $\lambda_E > 0$,[15]

$$\frac{dE_P}{dV_P} = \frac{1}{2\lambda_E}. \tag{20}$$

[15] See Markowitz (1987, Ch. 7) for a discussion of the CLA for the step in which $\lambda_E$ actually reaches zero.

### EXHIBIT 3
**Geometry of the EGP Algorithm**




Each portfolio corresponds to a point at which a line with slope $2\lambda_E$ is tangent to the efficient frontier. As the slope of this line varies from infinity down to zero, the tangent point traces out the entire efficient frontier from its high-return, high-risk extreme down to $(E_{min}, V_{min})$.

Unlike the CLA, which finds all corner portfolios by varying the slope term $\lambda_E$, the EGP algorithm finds all corner portfolios by varying an intercept term $R_f$. In the absence of a risk-free security, EGP defines a parameter,

$$\lambda = R_f - R_f^0,$$

where $R_f^0$ is an intercept term for which the algorithm has already determined an optimal portfolio, that is, it is the intercept term corresponding to the previous corner portfolio.

The relationship between $\sigma_p = \sqrt{V_P}$ and $E_P$ along the efficient frontier is shown in Exhibit 3.

The curved line in the exhibit represents the efficient frontier itself. The EGP algorithm traces out the efficient frontier by finding each tangency point corresponding to a particular value of the intercept term $R_f$ as it increases from some preset minimum value up to a preset maximum value. The relationship between the parameter $R_f$ and the corresponding tangency portfolio is illustrated in Exhibit 3. From the geometry of Exhibit 3, it must be true for any $(E_P, \sigma_P)$ pair along the efficient frontier that

$$\frac{dE_P}{d\sigma_P} = \frac{E_P - R_f}{\sigma_P}. \tag{21}$$

Now, because $V_P = \sigma_P^2$, where $\sigma_P$ is the standard deviation of the portfolio's return, we have

$$\frac{dE_P}{dV_P} = \frac{dE_P}{d\sigma_P}\frac{d\sigma_P}{dV_P} = \frac{dE_P}{d\sigma_P}\frac{1}{2\sigma_P}.$$

Using (21), this becomes

$$\frac{dE_P}{dV_P} = \frac{E_P - R_f}{\sigma_P}\frac{1}{2\sigma_P} = \frac{E_P - R_f}{2V_P}. \tag{22}$$

Equating the derivatives in (20) and (22), we obtain

$$\frac{1}{\lambda_E} = \frac{E_P - R_f}{V_P}.$$

This is true for any $(E_P, V_P)$ pair along the efficient frontier. Therefore, in particular, it must be true for the pair $(E_{min}, V_{min})$, so we find that

$$R_f = E_{\min} - \frac{V_{\min}}{\lambda_E},$$

showing that a constant relationship exists between $R_f$ (the parameter varied in the EGP algorithm) and $\lambda_E$ (the parameter varied in the CLA). Thus, we have unified the CLA and the EGP algorithm.

## MARKET SIMULATION

One of our longer-term initiatives with Harry was to design and build a simulator that could explain the behavior of the market better than existing models. Harry often thought of himself as an "operations research kind of guy." In this aspect, he and Bruce had something in common beyond their explorations into portfolio theory. Bruce had studied operations research and, like Harry, had once worked for the Rand Institute. Harry's work with Rand in California in the 1950s produced the SIMSCRIPT programming language. Bruce's later work with the New York City Rand Institute was related to the development of a large-scale simulation model to optimize response times for the New York City Fire Department. The quest for a market simulator would be an opportunity for both to incorporate computer science and mathematical programming (optimization) into their work and to apply it to a real-world problem.

Many market models use continuous-time methods (such as those used in the Black–Scholes–Merton option-pricing model that formed the basis of portfolio insurance). These models may use assumptions—for example, that the underlying security price process is fixed and that prices change randomly and continuously over time. The models may be useful because they can often be solved analytically. They are not useful, however, when investment actions or changes in the underlying environment alter the price process. Nor may they tell us whether theories about the behavior of investors can explain the observed phenomena of the market.

We developed a market simulator, the Jacobs Levy Markowitz Market Simulator (or JLMSim) that has the potential to address these problems. JLMSim is an asynchronous, discrete-time, dynamic market simulator whose objective is to model the evolution of market prices and trading volumes over time. It assumes that price changes reflect events, which can unfold in an irregular fashion. The price process of securities is not fixed but is the result of simulated market participants trading with one another in order to maximize their own individual utility functions as conditions change and as random money flows occur into or out of the market. JLMSim allows users to model financial markets using their own inputs about the numbers and types of investors, traders, securities, and other entities that would have a bearing on markets in the real world.

Asynchronous models such as that used in JLMSim may also be better than continuous-time models for analyzing whether micro theories about investor behavior can explain market macrophenomena. From time to time, the market manifests liquidity black holes, which seem to defy rational investor behavior. One extreme case was the stock market crash on October 19, 1987. When prices fell precipitously and discontinuously on that day, rational value investors could have stepped in to pick up bargain stocks, but few did. Asynchronous models could explain both the abundance of sellers and the dearth of buyers. Our experiments with JLMSim showed that a relatively small proportion of momentum investors can destabilize markets, overwhelming value investors. Similarly explosive behavior can result when traders do not anchor their bid or offer prices to existing market prices. Our belief is that an asynchronous-time market simulator—such as JLMSim, capable of modeling the agents and market mechanisms behind observed prices—is much better than continuous-time models at representing the reality of markets.

So far, we have described JLMSim running in its dynamic analysis (DA) mode to simulate market behavior. More details about JLMSim running in the DA mode are given in Jacobs, Levy, and Markowitz (2004, 2010). JLMSim can also operate, in what we call capital markets equilibrium (CME) mode, to seek equilibrium expected returns, as we describe in the following.

Black and Litterman (1992) suggested a "reverse optimization" procedure to find equilibrium expected security returns that are consistent with a given covariance matrix and a specified market portfolio. The Black–Litterman (BL) procedure operates under the CAPM assumptions that investors can borrow all they want at the risk-free rate and that portfolios are constrained only by budget.

The BL procedure for estimating expected returns has the following inputs: a covariance matrix; percentages of the market portfolio invested in various securities; views about expected returns for some, all, or none of the securities; and a parameter that serves to anchor the general level of expected returns. If the user supplies no views, the BL procedure produces capital market equilibrium expected return estimates that would clear the market.

Under the BL assumptions, investors are essentially unconstrained and can borrow without limit at the risk-free rate. Under these assumptions, the Tobin (1958) separation theorem applies, and all investor portfolios lie on the straight capital market line (CML). Portfolios on the CML consist of various combinations of the riskless security and the *same* portfolio of risky securities.

In reality, contrary to the assumptions of the BL procedure, investors are constrained and cannot borrow without limit at the risk-free rate. Thus, investor portfolios do not all lie on the CML. Instead, they lie on the curved efficient frontier at positions determined by investor risk tolerances, and the compositions of the portfolios of risky securities differ from investor to investor. In such cases, the market portfolio may not even be efficient (see Markowitz 2005).

In CME mode, JLMSim seeks capital market equilibrium expected returns for markets in which the CAPM assumptions do not necessarily hold. It allows users to solve for expected returns for markets in which investors cannot borrow, or have restricted borrowing, and in which investors can or cannot short. In other words, it can be used to seek equilibrium expected returns for any of the large variety of markets that can be simulated by JLMSim. Naturally, not all such markets are consistent with equilibrium solutions. We have not explored the convergence properties of JLMSim for all such markets.

In CME mode, capital market equilibrium expected security returns are found by adjusting securities' expected returns, thereby causing investors to change their portfolios in such a way that the aggregate of all investors' portfolios converges to given (or target) market portfolio weights. Generally, if the weight of a security in the current market portfolio is above a given target weight, the simulator lowers the security's estimated expected return. If the current market weight is below the target weight, the simulator raises the security's estimated return.

To create a realistic representation of market participants' holdings when running JLMSim in CME mode, the user can provide several investor templates that would place representative portfolios on various parts of the efficient frontier and not just on the CML. With such placement, the BL assumptions are no longer satisfied. Therefore, the BL procedure would not provide correct equilibrium expected returns.

In contrast, JLMSim does provide correct results under these circumstances. The estimated equilibrium expected returns at the end of a CME run are the returns that are consistent with the given market portfolio and given covariance matrix. Furthermore, they are consistent with realistic assumptions regarding limits on investors' ability to borrow. Specific examples are provided in Jacobs, Levy, and Markowitz (2010).

Harry, in his foreword to Guerard (2010), distinguished between the Jacobs–Levy–Markowitz asynchronous discrete event simulation and the Sharpe single-period model and the Merton continuous-time models:

> Sharpe (1964) and Lintner (1965) present an "equilibrium" model. They say that, given certain assumptions, "in equilibrium" such-and-such will be true. Their model may be interpreted as a single-period or a static steady-state model. On the other hand, Merton (1990) and his many followers present continuous-time models in which price is assumed to follow one or another stochastic process, assumed a priori. In contrast to both these types of models—the static and the continuous-time dynamic—the model presented by Jacobs, Levy, and Markowitz (2004, 2010) is an asynchronous discrete event simulation in which time advances, usually in irregular jumps, to the next most imminent event. Prices are endogenous, resulting from the interaction of thousands of investors and their traders following various investment and trading rules. (p. E1)

We hope that over time and with input from the finance community, JLMSim will develop into a simulator that researchers can use to create realistic dynamic models of the market. Potentially, these models could help to test the effects on securities' prices of real-world events such as changes in investment strategy or regulatory policy. Other examples may include examining the effects on markets of various levels of passive portfolio management or leverage, or investigating the impact of institutional structures (such as minimum tick sizes or the use of crossing networks) or regulatory policies (including, e.g., capital gains taxation and circuit breakers). JLMSim can already be used to compute capital market equilibrium returns under fairly realistic constraints that would make the same problem analytically intractable.[16]

## RISKS OF PORTFOLIO LEVERAGE

Our work on long–short portfolios led us to consider how leverage affects portfolio risk and choice. To the extent that leverage increases a portfolio's volatility, mean–variance optimization captures some of the risk associated with leverage, but it fails to capture other components of risk that are unique to using leverage. A portfolio with leverage differs in a fundamental way from one without leverage. A leveraged investor must take into account the risks and costs of margin calls, which can force borrowers to liquidate securities at adverse prices due to illiquidity, the possibility of losses exceeding the capital invested, and even bankruptcy. In extreme cases, the adverse consequences of leverage can impact the stability of markets, as in 2008, when the highly leveraged housing sector collapsed, taking down the debt instruments that supported it and precipitating a global financial crisis (Jacobs 2009).

When we wrote our first paper on this topic (Jacobs and Levy 2012), Bruce called Harry to tell him that we expanded the mean–variance model to account for leverage risk. Harry was skeptical.

"When you take on leverage, the portfolio's volatility increases, and that increase is taken into account with the volatility term in the model," he said.

"Yes," Bruce replied, "but when you leverage a portfolio, you're indebted to the prime broker, and if your account experiences losses, there will be a margin call."

"I see," Harry responded.

Harry's belief may have stemmed from the conclusions of Kroll, (Haim) Levy, and Markowitz (1984), which assumed a proportional, or linear, increase in portfolio volatility with portfolio leverage. In a section of the paper entitled "The Effect of Leverage," they state: "Leverage increases the risk of the portfolio. If the investor

[16] Those interested in finding out more about JLMSim, or experimenting with it, can download it from https://jlem.com/research#/market-simulation/5,/selection/1. Since we made it available, the simulator has been used by researchers in more than 70 countries.

borrows part of the funds invested in the risky portfolio, then the fluctuations of the return on these leveraged portfolios will be proportionately greater."[17]

There was precedence in the literature for the linearity assumption. Hester (1967, p. 42) simplified his efficient frontier calculations by assuming "investors believe that they will incur a margin call with probability zero," or alternatively, "the investor retains other assets which he may use to offset margin calls." He acknowledged this was the least palatable of his assumptions. He concluded that the "Markowitz efficient frontier portfolio locus is dominated by a locus which allows short sales and margin positions" (p. 50). Later, Pogue (1970) had the same finding regarding the efficient frontier in an extended Markowitz model that includes shorts and leverage as well as transactions' costs and taxes. He, too, recognized the possibility of margin calls (p. 1014) in which the "borrower [has to] increase the equity status of his account." In his footnote 24 (p. 1020), he recognized that the provision of credit will depend on the creditor's risk aversion. However, in both articles, investor aversion to leverage risk was not modeled.

Harry responded to our proposal for a mean–variance-leverage model, and in particular to our 2013 article "Leverage Aversion, Efficient Frontiers, and the Efficient Region" (2013a), with his own solution for optimization with leverage risk. In "How to Represent Mark-to-Market Possibilities with the General Portfolio Selection Model" (Markowitz 2013), he suggested extending the general model by including a measure of short-run volatility, as determined by a stochastic margin call model. As we responded in turn (2013b), however, a stochastic margin call model has yet to be developed, whereas the mean–variance-leverage model is available for immediate use.

Jason Zweig (2012) of the *Wall Street Journal* quotes Bruce and Harry:

> "Conventional portfolio theory says not to hold all your eggs in one basket," says Mr. Jacobs. What that misses, he adds, is that "using leverage is like piling baskets of eggs on top of one another until the pile becomes unsteady." Borrowed money can make an optimally diversified—and theoretically "safe"—portfolio risky.
>
> Prof. Markowitz agrees. If you're a diversified investor who can afford to be patient, you should worry primarily about how you'll do on average in the long run, he says.
>
> "But if you're leveraged, then you can get wiped out before the long run comes," he says. Keeping that in mind as you diversify, he adds, is "very important."

There can be another benefit to adding a leverage-aversion cost term to the objective function. In the earlier section "Integrated Long–Short Optimization and Portfolio Constraints," we noted that Harry said adding constraints to a mean–variance optimization may produce a portfolio that provides higher expected utility. This modification of the optimization problem can mitigate its sensitivity to uncertainty. Such "regularization" can be achieved by adding portfolio constraints or a penalty term to moderate the effects of extreme outcomes. Regularization is often interpreted as a form of robust optimization (e.g., Boyd et al. 2024).

Harry always believed that theories, including his own, are improved by incorporating the innovations of others. He agreed to write a new foreword (Markowitz 2017) for the second edition of our *Equity Management* (2017) book, in which he briefly

[17] Note that linearity may hold under certain unrealistic conditions, such as unlimited capital availability, frictionless markets with continuous pricing, and costless liquidity which, in theory, could render margin calls of practical irrelevance.

reflected on our disagreement with all the wit and wisdom we had come to expect from him. He wrote: "Some of the new sections include works on which Jacobs, Levy, and I collaborated—or, in the case of leverage aversion, debated—so, we have continued to build upon each other's research."

### Implications of Leverage Risk

Mean–variance analysis will result in optimal unleveraged (long-only) portfolios for investors not able to tolerate any leverage. But, for investors who use leverage, mean–variance analysis can result in "optimal" portfolios that are highly leveraged. This is because mean–variance optimization implicitly assumes the investor has an infinite tolerance for the unique risks of leverage. In practice, however, investors *are* leverage averse. If offered a choice between a portfolio having a particular expected return and variance *without* leverage and another portfolio that offers the same expected return and variance *with* leverage, most investors would prefer the former portfolio. The conventional mean–variance utility function cannot distinguish between these two portfolios because it does not account for an important aspect of investors' behavior, namely, investors' aversion to the unique risks of leverage.

Because investors are typically leverage averse, however, those who use leverage usually limit it. They often do so in a largely *ad hoc* manner, choosing a leverage level (often dependent on the risk of the underlying securities) and imposing it on the portfolio by means of a leverage constraint in the optimization process.[18] In Jacobs and Levy (2013a), we propose an alternative solution that involves augmenting portfolio theory and mean–variance optimization by incorporating a term for investor leverage tolerance.[19]

The resulting mean–variance-leverage model provides the utility of a leveraged portfolio for a leverage-averse investor:

$$U = \alpha_P - \frac{1}{2\tau_V}\sigma_P^2 - \frac{1}{2\tau_L}\sigma_T^2\Lambda^2, \tag{23}$$

where $\alpha_P$ is the expected active return (relative to the benchmark) of the leveraged portfolio; $\sigma_P^2$ is the variance of the leveraged portfolio's active return; $\tau_V$ is the investor's risk tolerance with respect to the variance of the portfolio's active return, which we will refer to as volatility tolerance; $\sigma_T^2$ is the variance of the leveraged portfolio's total return; $\tau_L$ is the investor's leverage tolerance; and $\Lambda$ is the portfolio's leverage, defined as

$$\Lambda = \sum_{i=1}^{N} |h_i| - 1, \tag{24}$$

where $h_i$ is the portfolio holding weight of security $i$ for each of the $N$ securities in the selection universe.[20]

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[18] Markowitz (1959) showed how to use individual security and portfolio constraints in MV optimization.

[19] Augmenting the mean–variance paradigm is robust given its universality, see Benveniste, Kolm, and Ritter (2024).

[20] When the investor's leverage tolerance is zero, portfolio leverage, $\Lambda$, will be zero. Note that because short positions entail unlimited liability, they, like leveraged long positions, expose the portfolio to losses beyond the invested capital. Hence, investors with zero leverage tolerance would impose a no-shorting constraint on the portfolio.

We assume that investors have the same aversion to leveraged long positions as they do to short positions; however, this assumption may not be the case in practice because short positions have

We use a squared term for leverage so that both risk components would have similar functional forms. We use the portfolio's total volatility as a multiplier because the unique risks of leverage relate more to a portfolio's total volatility than to the volatility of its active returns. That is, the risk that portfolio losses will trigger a margin call or exceed the capital invested depends on the portfolio's total volatility. The leverage tolerance term assumes that the risks of leverage rise with the product of the variance of the leveraged portfolio's total return and the square of the portfolio's leverage. Furthermore, this leverage dimension of risk will not be constant, but will vary across different portfolios having different volatilities.

To investigate this utility function, we looked at illustrative ranges for the volatility and leverage tolerance terms. As one reference point, a value of $\tau_V = 0$ corresponds to an investor who is completely intolerant of active volatility risk. Such an investor would choose an index fund. As another reference point, a value of $\tau_V \approx 1$ causes quadratic utility of return to be equivalent to log-utility of wealth, a utility function often used in the finance literature (Levy and Markowitz 1979). Thus, we chose $\tau_V \in [0, 2]$. For illustrative purposes, we chose $\tau_L$ to span the same range as $\tau_V$.

If $\alpha_i$ is the expected active return of security *i*, $b_i$ is the weight of security *i* in the benchmark, $x_i$ is the active weight of security *i* (and by definition $x_i = h_i - b_i$), $\sigma_{ij}$ is the covariance between the active returns of securities *i* and *j,* and $q_{ij}$ is the covariance between the total returns of securities *i* and *j,* then Equation (23) can be written as

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} h_i q_{ij} h_j\right)\Lambda^2. \tag{25}$$

Using Equation (24), and because $h_i = b_i + x_i$, Equation (25) becomes

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N}\sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} (b_i + x_i) q_{ij} (b_j + x_j)\right)\left(\sum_{i=1}^{N} \left|b_i + x_i\right| - 1\right)^2. \tag{26}$$

Equation (26) is the utility function to be maximized expressed in terms of active security weights.

### Mean–Variance-Leverage Model versus Mean–Variance Model

The mean–variance (MV) model is a special case of the mean–variance-leverage (MVL) model. As the investor's tolerance for the unique risks of leverage approaches zero, the investor has an infinite aversion to leverage, and the optimizer forces the portfolio's leverage level to zero. The MVL model reduces to the traditional long-only MV model. At the other extreme, as the investor's tolerance for the unique risks of leverage approaches infinity, the investor has no aversion to leverage. The leverage term in the MVL model is multiplied by zero leverage aversion, and that term drops out of the MVL utility function. Again, the MVL model reduces to the MV model.

The MV model, used with a constraint enforcing zero leverage, therefore implies that the investor has an infinite aversion to the unique risks of leverage, or zero leverage tolerance. Used without a leverage constraint, the MV model implies that the investor has zero aversion to the unique risks of leverage, or infinite leverage tolerance. Note that, although we observe zero leverage tolerance in practice—some investors are averse to any borrowing—infinite leverage tolerance seems contrary to

potentially unlimited liability and are susceptible to short squeezes. One could model the aversion to long and short positions asymmetrically. Because doing so would complicate the algebra, for simplicity, we used a common leverage tolerance.

investor behavior because it can give rise to extreme levels of leverage in the absence of upper bounds on individual security holdings (Jacobs and Levy 2013a, 2014a).

To avoid excessive leverage, the common practice is to constrain it at some level. For an investor who is averse to leverage, however, using the conventional MV utility function and optimizing with a leverage constraint is unlikely to lead to the portfolio offering the highest utility. This is because a leverage constraint denies the investor the ability to balance the economic tradeoffs between expected portfolio return, portfolio volatility risk, and portfolio leverage risk (Jacobs and Levy 2014a).[21]

In Markowitz (2000), Harry wrote: "By and large, I still believe, as I did in 1952, that mean–variance analysis can provide the 'right kind' of diversification for the 'right reason.' Diversification makes sense, and proper diversification depends on a consideration of covariances." The mean–variance-leverage model is an approach that allows investors to determine the "right amount" of portfolio leverage with the "right kind" of diversification.

## A Practical Application of the MVL Utility Function: 130–30 Long–Short

To examine the effects of different levels of leverage tolerance, we applied the MVL utility function to an enhanced active equity (EAE), or 130–30–type, long–short portfolio structure. For expository purposes, we assume the strategy is self-financing and entails no financing costs. An enhanced active 130–30 portfolio, for instance, has leverage of 60% and an enhancement of 30%.

We found EAE portfolios that maximize the utility function represented by Equation (26) for a range of volatility and leverage tolerance pairs $(\tau_V, \tau_L)$, subject to standard constraints. The standard constraint set for an EAE portfolio is as follows.

$$\sum_{i=1}^{N} h_i = 1.$$

This full-investment (net longs minus shorts) constraint requires that the sum of the signed holding weights equals 1.

$$\sum_{i=1}^{N} h_i \beta_i = 1.$$

This beta constraint (where $\beta_i$ is the beta of security $i$ relative to the benchmark) requires that the portfolio's beta equals 1.

In terms of active weights, these constraints are expressed as follows.

$$\sum_{i=1}^{N} x_i = 0.$$

The sum of security active underweights relative to benchmark (including short positions) equals the sum of security active overweights.

$$\sum_{i=1}^{N} x_i \beta_i = 0.$$

The sum of the products of security active weights and security betas equal zero.

[21] Edirisinghe, Chen, and Jeong (2023) extended MVL to also include liquidity costs in their MVLL model.

In addition to these standard constraints, we constrained each security's active weight to be between –10% and +10%.

While MV optimization is a quadratic mathematical problem, MVL optimization is a quartic problem. Our solution method to maximize the MVL utility function was to use fixed-point iteration that applied a quadratic solver iteratively (Jacobs and Levy 2013b). Rewriting Equation (26) as the following set of two equations,

$$U = \sum_{i=1}^{N} \alpha_i x_i - \frac{1}{2\tau_V} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j - \frac{1}{2\tau_L} \sigma_T^2 \left( \sum_{i=1}^{N} |b_i + x_i| - 1 \right)^2$$
$$\sigma_T^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} (b_i + x_i) q_{ij} (b_j + x_j), \tag{27}$$

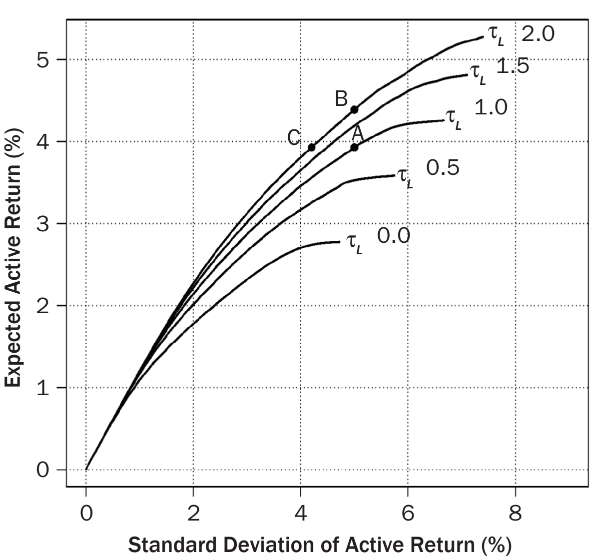
we chose an initial estimate of $\sigma_T^2$, and used this as a constant to maximize the utility function. This maximization provided estimates of the $x_i$s, which were used to compute a new estimate of $\sigma_T^2$ using the second equation in Equation set (27). With the new estimate of $\sigma_T^2$, we repeated the optimization to find new estimates of the $x_i$s. This iteration was repeated until successive estimates of $\sigma_T^2$ differed by a *de minimis* amount.

Using data for stocks in the S&P 100 Index over the two years ending September 30, 2011, and the estimation procedures described in Jacobs and Levy (2012), we derived optimal portfolios given a range of leverage and volatility tolerances.[22] Exhibit 4 illustrates, in a familiar two-dimensional volatility risk–return framework, how consideration of leverage aversion can affect the investor's choice of the optimal portfolio.

The exhibit displays efficient frontiers for levels of volatility and leverage tolerance ranging from 0 to 2. For each curve, leverage tolerance remains constant, at a level of 0, 0.5, 1.0, 1.5, or 2, while the investor's volatility tolerance increases along the curve from 0 at the origin to 2. The 0 leverage tolerance curve represents an investor unwilling to use leverage, that is, an investor who prefers long-only portfolios. As the exhibit shows, increasing leverage tolerance allows higher efficient frontiers. (Note that because different security active weight constraints become binding as one moves along each of the constant leverage-tolerance frontiers, a curve connecting the endpoints would not be smooth.)

## EXHIBIT 4

**Efficient Frontiers for Various Leverage Tolerance Cases**




It might at first appear that investors would prefer the highest level of leverage obtainable, as it offers the highest return per unit of risk. However, when leverage tolerance is considered, it becomes apparent that each frontier consists of the set of optimal portfolios for an investor with the given level of leverage tolerance.

For example, consider the three portfolios represented by the points labeled A, B, and C in Exhibit 4; their characteristics are provided in Exhibit 5. Portfolio A is the optimal portfolio for Investor A, who has a leverage tolerance of 1 and a volatility tolerance of 0.24. This is a 125–25 portfolio with a standard deviation of active return of 5% and an expected active return

[22] While for expository purposes we estimated the variance of the portfolio's total return based on historical data in the same way that we estimated the variance of the portfolio's active return, the investor in practice could estimate these variances on a forward-looking basis, taking into account security position sizes relative to the market and the expected market impact upon liquidation. Note that leverage increases portfolio illiquidity. Leverage and illiquidity are different, however, because illiquid portfolios without any leverage are not exposed to margin calls and cannot lose more than the capital invested.

of 3.93%. The utility for Investor A, $U_A$, of Portfolio A is 2.93. Both Portfolio B and Portfolio C dominate Portfolio A in an expected-active-return–standard-deviation framework; Portfolio B offers a higher expected active return (4.39%) at the same standard deviation (5%), while Portfolio C offers the same level of expected active return (3.93%) at a lower standard deviation (4.21%). But, for an investor with leverage tolerance of 1 (Investor A), Portfolio A offers a higher utility than that of Portfolio B or Portfolio C. For Investor A, both Portfolio B (139–39) and Portfolio C (135–35) have too much leverage.

Conventional MV optimization and efficient frontier analysis are inadequate to determine optimal portfolios when investors use leverage and are averse to leverage risk. They fail to recognize that most investors are willing to sacrifice some expected return in order to reduce leverage risk, just as they sacrifice some expected return in order to reduce volatility risk.

Exhibit 6 illustrates the efficient frontiers *without* the active weight constraints for various levels of investor leverage tolerance and those for various levels of investor volatility tolerance. Every leverage-tolerance level has a corresponding two-dimensional MV efficient frontier. Similarly, for a particular level of volatility tolerance, there is a corresponding two-dimensional MV efficient frontier. Because Exhibit 6 assumes no constraint on the security active weights, the curve linking the optimal portfolios for an investor with a leverage tolerance of 2 is smooth (unlike in Exhibit 4).

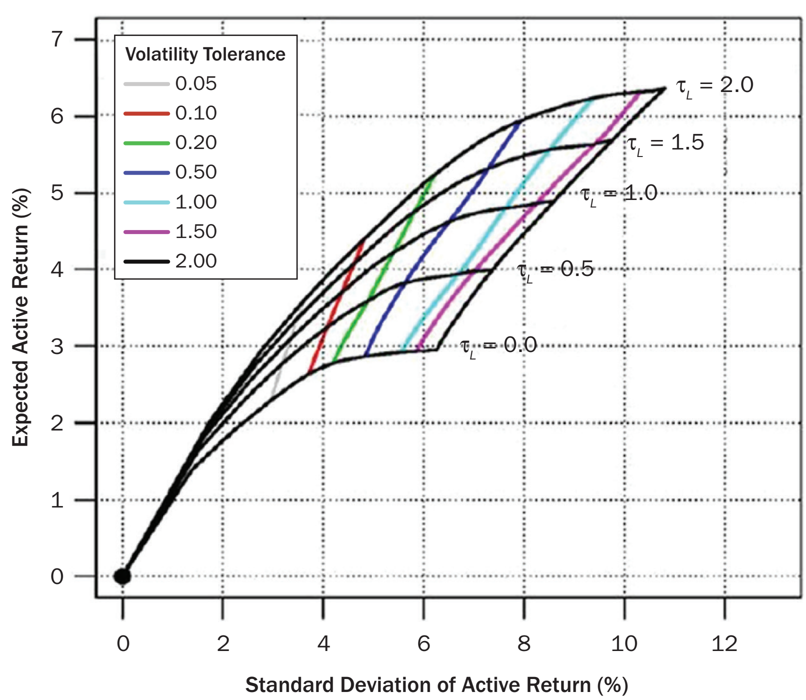
Furthermore, without the security active weight constraints, both the standard deviation of active return and the expected active return range are higher than in Exhibit 4. As either volatility tolerance or leverage tolerance declines from 2, the frontiers shift to the left and downward. When volatility tolerance is zero, the optimal portfolio—an index fund—lies at the origin.

## EXHIBIT 5
### Portfolio Characteristics

| | $\tau_L$ | $\tau_V$ | EAE* | $\sigma_P$ | $\alpha_P$ | $U_A$ |
|---|---|---|---|---|---|---|
| A | 1.00 | 0.24 | 125-25 | 5.00 | 3.93 | 2.93 |
| B | 2.00 | 0.14 | 139-39 | 5.00 | 4.39 | 2.72 |
| C | 2.00 | 0.09 | 135-35 | 4.21 | 3.93 | 2.68 |

**NOTE:** *Rounded to the nearest percentage.

## EXHIBIT 6
### Efficient Frontiers for Various Leverage ($\tau_L$) and Volatility ($\tau_V$) Tolerance Cases with No Security Active Weight Constraint



Depending on an investor's leverage and volatility tolerances, the optimal portfolio will lie somewhere in the MVL *efficient region* shown. Once again, the critical roles of both leverage and volatility tolerance in portfolio selection are apparent.

Because the efficient frontier differs for investors with different tolerances for leverage, MVL optimization must be used to solve for optimal portfolios that now lie on a three-dimensional MVL efficient surface (Jacobs and Levy 2014b), shown in Exhibit 7. When leverage aversion is included, lower MVL efficient frontiers having less leverage are generally optimal. This choice of a lower frontier seems to contradict the basic tenets of portfolio theory. The investor's preference for a lower frontier despite its lower expected returns, however, reflects the investor's aversion to the unique risks associated with the higher frontier's higher leverage. The optimal frontier for a particular investor depends upon the investor's leverage tolerance, and the optimal portfolio for that investor on *that* frontier depends upon the investor's volatility tolerance. Both volatility tolerance and leverage tolerance play critical roles in portfolio choice.

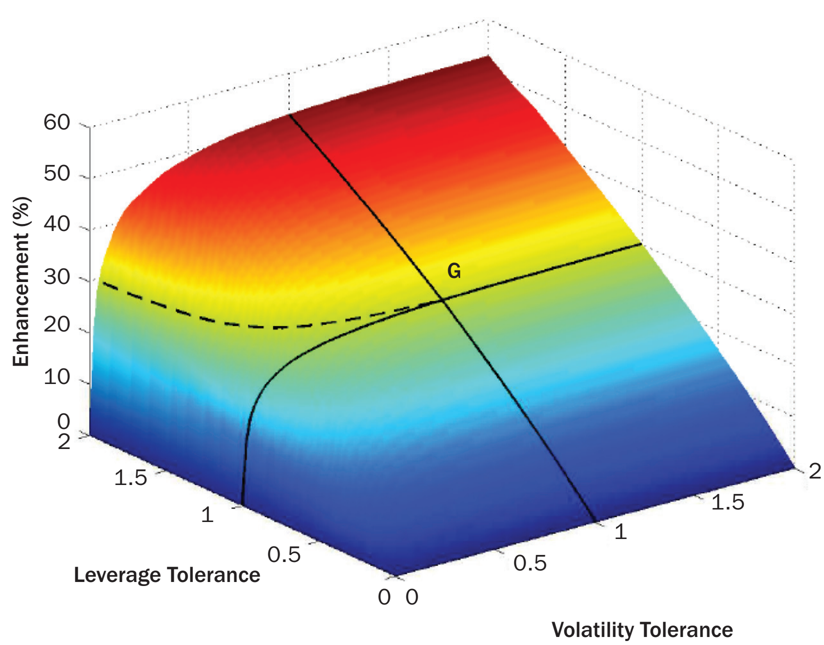
To estimate an investor's own volatility and leverage tolerances, the investor could select different portfolios from the efficient surface, run a Monte Carlo simulation that generates a probability distribution of ending wealth for each portfolio, and then infer their volatility and leverage tolerances based on their preferred ending wealth distribution. Alternatively, investors could use asynchronous simulation, which can account for the occurrence of margin calls, including security liquidations at adverse prices (Jacobs, Levy, and Markowitz 2004, 2010).

For investors with volatility and leverage tolerances of 1 in our example, as shown in Exhibit 7, the optimal portfolio enhancement, at point G, is about 30% for a 130–30 long–short portfolio. For investors with greater (lesser) tolerances, the optimal portfolio enhancement will be greater (lesser).

The MVL model allows the investor to determine the optimal portfolio for any combination of volatility tolerance and leverage tolerance. It shows that an investor's level of leverage tolerance can have a large effect on portfolio choice.

## EXHIBIT 7

**Mean–Variance-Leverage Efficient Surface**



## CONCLUSION: THE THEORY, PRACTICE, AND FUTURE OF INVESTING

Harry's analytical and computational ideas on portfolio theory were originally met with skepticism in the investment industry. After all, in the 1950s, most investors focused on stock picking, and computer power was scarce and expensive. Critics claimed his theories could not be translated into practice.

Yet Harry's portfolio theory eventually gained wide acceptance and remains used and useful. One reason for the theory's longevity is its adaptability to practice. As Harry noted in his foreword (Markowitz 2000) to the first edition of our book, *Equity Management*, "mean–variance analysis should not be considered a black box that can be set on automatic and allowed to run portfolios on its own." Human judgment is critical, and theory must be shaped by real-world considerations.

Harry remained intrigued by the workings of financial markets into his eighties and nineties. We lost him at age 95, but his theories and influence on practice live on. His wife Barbara would ask, "Harry, when do you plan to retire?" Harry would respond, "When I do retire, I'd want to do something I really enjoy. And that's what I'm doing now, every day—playing in my sandbox."

Bill Sharpe once said about Harry: "Ordinary people think about problems; extraordinary people think about how to think about problems."

Modern portfolio theory is not simply a solution to a problem. It is a revolutionary way of thinking about the problem of investment uncertainty. From it evolved the vast field of quantitative finance that has produced extraordinary innovations for 70 years. Not a small part of the workings of the market today reflects Harry's ideas.

It is fitting to conclude with one of Harry's favorite expressions: "Let's write it up and give it to the world."

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